Quantum Optics. Lections 2009.

I. Introduction

(Teilgebiet)
Quantum Optics is a subsection of Quantum Electrodynamics, QED.

Quantumeigenschaften
The aim of QED is to describe the eigenstates (quantum)
(Strahlungfeldes) Wechselwirkung
of Electromagnetic radiation fields and their interaction
Materie unterteilt
with matter. QED can be subdevided in following topics

Zweig
Ⓐ High energy QED (the Branch of Elementary Particle Physics)

$\varepsilon \gtrsim mc^2$

- relativistic covariant formulation of a field theory for a system of radiation fields and matter, (d.h.) e.g.
  - Radiation field with covariant quantization in Lorenz (Eichung) gauge
  - Relativistic description of the quantum theory of matter (freiheitsgrade) degrees of freedom, for example using Dirac or Klein-Gordon theory.

- Phenomena, that cannot be described without high energy [illegible], are e.g.
  - (Abweichung) Deviation of the gyromagnetic constant (g-factor) of electron from 2 in vacuo.
  - Generation of new particles by the field of high energy, for example ~~by~~ electron-positron Pairs.

Photons / atoms 2.2.02.686
Introduction to quantum electro
Claude Cohen-Tannoudji
Jacques Dupont-Roc
Gilbert Grynberg

C. Gerry P Knight
Introductory Quantum
[illegible]

Ⓑ Lower Energy QED (Quantum Optics)

- Description of eigenstates (quantum) of radiation field in
  Bereich
  "Optical range" of electromagnetic spectrum. This
  Wellenlängen
  is defined as the range of wave lengths (in vacuo)

Infrared IR : $8\cdot10^{-7}_{m} \leq \lambda \leq 10^{-4}m$
sichtbar
visible light : $4\cdot10^{-7} \leq \lambda \leq 8\cdot10^{-7}m$

(frequency approximately $f = \frac{1}{T} \approx 500THz$
$T \approx 1fs = 10^{-15}sec$)

ultraviolet UV : $4\cdot10^{-7}m > \lambda > 10^{-9}m$

- entspricht
  For this range energy of the field corresponds to
  Anregunsenergie
  typical excitation energy of condensed matter.
  Ruheenergie
  Howerer! : usually $E_{field} \ll mc^2$ (mc^2 is restenergy
  of particles). Therefore, effects of positron-electron
  erzergung vernachlässichbar
  pair creation are negligible.

- Formulation of quantum field theory ~~in~~ using non-covariant version, for example using Coulomb gauge version

- For such energy range particles ~~of~~ matter can be considered non-relativistic and one can use standard Schrödinger quantization theory.

- Topics of quantum optics are, for example
  - Quantum nature of light
    Hohlraumstrahlung
    (Plank description of black body radiation is the starting point of Quantum theory)
  - Spontaneous emission of light
  - unschärfen
    Effects of quantum mechanics uncertainty on measurements of electromagnetic fields
  - zustände
    Signatures of non-classical light field states e.g. "Antibunching" in measurements of intensity correlations
  - Discussion of fundamental questions of Quantum mechanis for example stochastic properties of Quantum theory.
    Keyword: optical realization for Einstein-Podolsky-Rosen Paradox
  - maßgeschneidert
    States with artificially created uncertainty "squeezed states"
  - Coheren light states and Quantum fluctuations

II Quantization of radiation fields

Quanization programm

a) Classical description of the field – Maxwell equation. Formulation with help of canonic conjugate ~~operators~~ variables

b) canonical conjugate variables substituted by pairs of non commuting operators

→ Quntization procedure is analog of quantization of particles in potential. New aspects of the field quantization:

- Radiation fields have infinitely many degrees of freedom
  unendlich viele — freiheits grade
- Maxwell equation has too many "redundant" variables E and B have six components, while A has 4.
- Vector potential A is gauge invariant, this further reduces the number of relevant degrees of freedom.

One must write Maxwell equation in a form, suitable for a simple quanization procedure

Trennung Wirbel Quellen

III 1. Separation of rotational and source fields.

Classical equation of fields – Maxwell equation

Strom Ladung

with current and charges

SI

$$\text{rot}\,\bar{E} = -\dot{\bar{B}} \qquad (1a)$$

$$\text{rot}\,\bar{B} = \varepsilon_0\mu_0\dot{\bar{E}} + \mu_0\bar{j} \qquad (1b)$$

$$\text{div}\,\bar{E} = \varepsilon_0^{-1}\rho \qquad (1c)$$

$$\text{div}\,\bar{B} = 0 \qquad (1d)$$

Gaussian units, SGS

$$\text{rot}\,\bar{E} = \frac{1}{c}\dot{\bar{B}}$$

$$\text{rot}\,\bar{B} = \frac{1}{c}\dot{\bar{E}} + \frac{4\pi}{c}j$$

$$\text{div}\,\bar{E} = 4\pi\rho$$

$$\text{div}\,\bar{B} = 0$$

---

offtopic ε_0, μ_0 – does not have physical meaning

Advantages of SGS $(\varepsilon_0\mu_0)^{-1} = c$ – light speed.

* B and E have the same dimensionality

* SGS – sm, g, sec SI – m, kg, sec + 1 amper

→ At this moment ρ and j are assumed given externally

Vector calculus, some facts

Any vector field that disappear at $r \to \infty$ or at some boundary (Randeines) of a closed space can unambiguesly (eindeutig) be written as a sum of two vectors

$$\bar{V} = \bar{V}_\omega + \bar{V}_Q \qquad (2)$$

so that

$$\text{div}\,\bar{V}_\omega = 0\,;\quad \text{rot}\,\bar{V}_\omega = \text{rot}\,\bar{V} \qquad (2a) \quad \overset{\text{Wirbel}}{\text{rotational field}}$$

$$\text{div}\,\bar{V}_Q = \text{div}\,\bar{V}_Q\,;\quad \text{rot}\,\bar{V}_Q = 0 \qquad (2b) \quad \overset{\text{Quellen}}{\text{Source field}}$$

From the uniquenes (eindeutigkeit) one has that once the boundary condition are fulfilled (erfüllt) and

$$\text{div}\,\bar{V} = 0 \qquad (3a)$$

$$\text{rot}\,\vec{V} = 0 \qquad (3b)$$

then

$$\vec{V} = 0 \qquad (3c)$$

Consequence for $\bar{E}$ and $\bar{j}$

$$\vec{E} = \vec{E}_W + \vec{E}_Q \qquad (4)$$

$$\vec{j} = \vec{j}_W + \vec{j}_Q \qquad (5)$$

Behauptung
Statement: Only $\vec{j}_W$ is coupled to $\vec{E}_W$

div{Eq (1b)} =>

$$0 = \text{div}\,(\varepsilon_0 \dot{\vec{E}} + \vec{j}) \underset{(2b)}{=} \text{div}\,(\varepsilon_0 \dot{\vec{E}}_Q + \vec{j}_Q) \qquad (6)$$

From def. (2b) => $\text{rot}\,(\varepsilon_0 \dot{\vec{E}}_Q + \vec{j}_Q) = 0$ (7)

Wegen
Because of Eq. (3) and Eqs (6)−(7)

$$\varepsilon \dot{\vec{E}}_Q + \vec{j}_Q = 0. \qquad (8)$$

From Eq(1d) $\vec{B}_Q = 0$ (9)

=> Maxwell equation separates into a rotational part and a source part

Rotational equations

$$\text{rot}\,\vec{E} = \text{rot}\,\vec{E}_W = -\dot{\vec{B}}_W \qquad (10a)$$

$$\text{rot}\,\vec{B} = \text{rot}\,\vec{B}_W = \mu_0 (\varepsilon_0 \dot{\vec{E}}_W + \vec{j}_W + \varepsilon_0 \dot{\vec{E}}_Q + \vec{j}_Q) =$$

$$\underset{(8)}{=} \mu_0 \varepsilon_0 \dot{\vec{E}}_W + \mu_0 \vec{j}_W \qquad (10b)$$

Source equations

$$\text{div}\,\vec{E} = \text{div}\,\vec{E}_Q = \varepsilon_0^{-1} \rho \qquad (11a)$$

$$\text{div}\,\vec{B} = \text{div}\,\vec{B}_Q = 0 \qquad (11b)$$

Result: Dynamics in variables

$\rho, \vec{j}_Q, \vec{E}_Q$ (Coulomb system)

vollständig
is fully decoupled from

$\vec{j}_W, \vec{E}_W, \vec{B}$ (Radiation fields system)

Discussion

- Because of Eq. (2) we obtained Eqs (11) which gives $\vec{E}_Q$ as a function of ρ at any given time instantane instantaneously and for an gesampten entire space. $\vec{E}_Q$ can be represented as

$$\vec{E}_Q = -\vec{\nabla}\cdot V_c \quad (\mathrm{grad}\, V_c)\ (V_c \text{ Coulomb potential})$$

Where
$$V_c(x,t) = \frac{1}{4\pi\varepsilon_0}\int d^3y\, \frac{\rho(y,t)}{|\bar{x}-\bar{y}|} \qquad (12)$$

Eq (8) [illegible]
$$\vec{j}_Q = -\varepsilon_0 \dot{\vec{E}}_Q$$
is then the definition of $\vec{j}_Q$ that relates $\vec{j}_Q$ with $\rho(x,t)$.

=> • $\vec{E}_Q$ at any Zeitpunkt time is fully described by charge distribution $\rho(x,t)$. $\vec{E}_Q$ contains no degrees of freedom, that can be used to describe ~~the~~ independent field dynamics, and it is fully described by quantum theory of matter (ρ).

- Aufgrund Due to zeitabteilungen time derivatives in ~~in~~ Eq. (10a) and Eq. (10b) $\vec{E}_W$ and $\vec{B}$. are fest gelegt connected with $\vec{j}_W$ and their Anfangswerten initial conditions $\vec{E}_W(x,t=t_0)$ and $\vec{B}(x,t=t_0)$. Consequently apart from $\vec{j}_W$, $\vec{E}_W$ and $\vec{B}$ contain "free" components. That free components darstellen define dynamical degrees of freedom ~~from~~ of $\vec{E}_W, \vec{B}$ which can be used for the quantization of the electromagnetic field.

- Decomposition into rotor and souce fields in Fourier space is equivalent to the decomposition to the transverce and longitudinal fields.

- The decomposition into longitudinal and transverse fields is not invariant with respect to Lorente transformation (can you ~~check~~ check this)

- The decomposition is not an approximation and thus does not contradicts relativistic principles. In particular Instanteneous coulomb field potential does not lead to instanteneous field interaction, $\vec{E}$ contains only retarded components.

- For some applications (for all in high energy physics) Lorentz covariance is important. Then for such theory a separation into $\vec{E}_w$ and $\vec{E}_Q$ is not convenient, $\vec{E}_Q$ should be considered also as a degree of freedom. Typical approach:
  - all four components of the vector potential must be quantized in Lorenz gauge.
  - Redundant degrees of freedom are considered with suitable ~~constraints~~ conditions that ~~put~~ ~~constrained~~ restrict the consideration to physically possible states.

For many (majority) applications in low enery range ~~because necessary~~ ~~than~~ a fully Lorentz covariant theory is not needed and will not be discussed.

II.2. Transverse field modes

II.2.1 Equation for the modes

In what follows we describe the ~~[illegible]~~ field degrees of freedom by $\vec{E}_W$ and B, while $\vec{E}_Q$ is assumed ~~[illegible]~~ defined externally by matter states.

Hereafter index "W" will be omitted, so we write

$$\mathrm{div}\,\vec{E} = \mathrm{div}\,\vec{B} = 0. \qquad (12)$$

We look for: A system of basis states (modes), that can be used to represent transverse fields and ~~re~~write the Maxwell equations

Ansatz: The modes are defined such that they
Substitution
are solutions to maxwel equations in Vacuum.

=> Separation of variables for a single mode:

$$\vec{E}(\vec{r},t) = \sqrt{\frac{\gamma}{\varepsilon_0}}\, q(t)\, \vec{e}(\vec{r}) \qquad (13a)$$

$$\vec{B}(\vec{r},t) = \sqrt{\gamma \mu_0}\, p(t)\, \vec{b}(\vec{r}) \qquad (13b)$$

(γ - is a normalisation constant)

Substitute in: $\mathrm{rot}\,\vec{E} = -\dot{\vec{B}}$; $\mathrm{rot}\,\vec{B} = \frac{1}{c^2}\dot{\vec{E}}$

this gives:

$$\mathrm{rot}\,\vec{e} = -\frac{\dot{p}}{q}\frac{1}{c}\vec{b} \qquad (14a)$$

$$\mathrm{rot}\,\vec{b} = \frac{\dot{q}}{p}\frac{1}{c}\vec{e} \qquad (14b)$$

Where $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

single mode - partial solution with satisfies boundary conditions

Separation constants

=> $$\omega_1 = -\frac{\dot p}{q} \qquad (15a)$$

$$\omega_2 = \frac{\dot q}{p} \qquad (15b)$$

23.04.09

These constants are not zero

If $\omega_1 = 0$, => rot $\vec e = 0$, div $\vec e = 0$ => $\vec e = 0$

rot $\vec b = 0$, div $\vec b = 0$ => $\vec b = 0$

similarly $\omega_2 \neq 0$.

From (15) obtain

$$\ddot q = -\omega_1 \omega_2 q \qquad (16)$$

=> Solutions $q(t)$ should be restricted (finite) at $-\infty < t < \infty$, then $\omega_1 \cdot \omega_2$ – is real and positive

=> $\omega_1 = |\omega_1| e^{i\varphi}$; $\omega_2 = |\omega_2| e^{-i\varphi}$

$p\gamma = \bar p$

p

Define: $\tilde\gamma = \left|\frac{\omega_2}{\omega_1}\right|^{1/4} e^{-i\varphi/2}$; $\omega = \sqrt{|\omega_1 \omega_2|}$

$$\tilde q = \frac{q}{\tilde\gamma}; \quad \tilde p = \tilde\gamma p; \quad \tilde{\bar e} = \tilde\gamma \bar e; \quad \tilde{\vec b} = \frac{\vec b}{\tilde\gamma}$$

$$\vec E = \sqrt{\gamma/\varepsilon_0}\, q \vec e = \sqrt{\gamma/\varepsilon_0}\, \tilde q \tilde{\vec e}$$

$$\vec B = \sqrt{\gamma \mu_0}\, p \vec b = \sqrt{\gamma \mu_0}\, \tilde p \tilde{\vec b}$$

=> $$-\dot{\tilde p}/\tilde q = -\tilde\gamma^2 \frac{\dot p}{q} = \tilde\gamma^2 \omega_1 = \omega \qquad (\cancel{17a})$$

=> $$\begin{cases} \dot{\tilde p} = -\omega \tilde q & (17a) \\ \dot{\tilde q} = \omega p & (17b) \end{cases}$$

Consequently two factors ω_1 and ω_2 are reduced to a single factor ω without changing $\vec E$ and $\vec B$.

In what follows we omit tilde "~" from the formulas for $\vec{e}, \vec{b}, p, q$.

=> Separation (13) lead to Eq (17) which is ~~a~~ canonical equations of motion for a harmonic oscillator with frequency ω.

Eq.(14) and (17) defines the modes for $\vec{e}$ and $\vec{b}$ as

$$\begin{cases} \operatorname{rot} \vec{e} = \frac{\omega}{c} \vec{b} & (18a) \\ \operatorname{rot} \vec{b} = \frac{\omega}{c} \vec{e} & (18b) \end{cases}$$

$$\operatorname{div} \vec{e} = \operatorname{div} \vec{b} = 0 \qquad (18c)$$

(follow from Eqs.(18a-b))

II.2.2 ~~Excitation~~ modes ~~and~~ with periodical boundary conditions

Eq. (18) for the modes gives different set of functions for different boundary conditions.

For example:
- Standing waves for the square box, normalized by volume "V".
- spherical waves for the spherical closed space, normalized by volume.

For the infinite space one can have both type of standing waves as a mode system. Both system can be used to decompose (characterise) an arbitrary transverse field in the entire space.

- The choice of a mode system is dictated by the prob Quite often it is convenient to use periodical conditions, where modes are also normalized by volume.

$$\Rightarrow \quad \vec{e}_{k,\lambda} = \vec{n}_{k,\lambda}\frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{r}} \qquad (19a)$$

$$\vec{b}_{k,\lambda} = \frac{ic}{\omega_k}\vec{k}\times\vec{n}_{k,\lambda}\frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{r}} \qquad (19b)$$

$V = L^3$; $\vec{k} = \frac{2\pi}{L}(j_x, j_y, j_z)$ (19c) j_i - are integer follows from perio city

$n_{k,\lambda}$ - are polarization vectors

The transversality is enforced by the condition

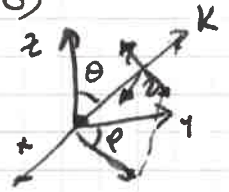
$$\vec{n}_{k,\lambda} \perp \vec{k} \qquad (19d)$$

$\Rightarrow$ For each $\vec{k}$ ~~there~~ there are two orthogonal polarization vectors $\vec{n}_{k,\lambda}$

$$(\vec{n}^*_{k,\lambda}\cdot\vec{n}_{k,\lambda'}) = \delta_{\lambda,\lambda'} \quad (19e) \qquad \omega_k = c|k| \quad (19f$$

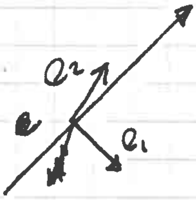
Ξ_w ~~It is useful to give a particular~~ example for polarization. One can take

$$\vec{K} = k \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (20)$$



- linear polarization

$$\vec{e}_{k,\lambda_1} = \begin{pmatrix} \cos\theta \cos\varphi \\ \cos\theta \sin\varphi \\ -\sin\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (21a)$$



$$\vec{e}_{k,\lambda_2} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad (21b)$$

- circular polarization

$$\vec{e}_{k,\lambda_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta\cos\varphi - i\sin\varphi \\ \cos\theta\sin\varphi + i\cos\varphi \\ -\sin\theta \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} (22a) \quad \frac{1}{\sqrt{2}}(e_{k1} + i e_{k2})$$

$$\vec{e}_{k,\lambda_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} i\cos\theta\cos\varphi - \sin\varphi \\ i\cos\theta\sin\varphi + \cos\varphi \\ -i\sin\theta \end{pmatrix} (22b) \quad \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}(ie_{k_1} +$$

Polarization vectors $\vec{e}_{k,\lambda_1}$ and $\vec{e}_{k,\lambda_2}$ together with $\vec{e}_{k,\lambda_3} = \frac{\vec{K}}{k}$ create an orthogonal basis.

So a vector $\vec{V}$ can be decomposed as

$$\vec{V} = \sum_{\ell=1}^{3} \vec{e}_{k,\lambda_\ell} \left(\vec{e}^{\,*}_{k,\lambda_\ell} \cdot \vec{V} \right)$$

=> in components $\vec{V}_i = \sum_{\ell=1}^{3} (\vec{e}_{k,\lambda_\ell})_i \sum_{j=1}^{3} (\vec{e}^{\,*}_{k,\lambda_\ell})_j V_j =$

$$= \sum_{j=1}^{3} \left\{ \sum_{\ell=1}^{3} (\vec{e}^{\,*}_{k,\lambda_\ell})_j (\vec{e}^{\,*}_{k,\lambda_\ell})_i \right\} V_j = \sum_j \delta_{ij} V_j$$

$$\delta_{ij} = \sum_{\ell=1}^{3} (\vec{e}^{\,*}_{k,\lambda_\ell})_j (\vec{e}_{k,\lambda_\ell})_i = \sum_{\lambda=1}^{2} (\vec{e}^{\,*}_{k,\lambda})_j (e_{k,\lambda})_i + \frac{k_i k_j}{k^2} \qquad (23)$$

II. 2. 3 Driven oscillations of the field

- Basis set of modes means, that not only vacuum field, a separate arbitrary solutions for transverse ME can be expanded ~~use~~ using that basis states

$$\begin{cases} E(\bar{r},t) = \sum\limits_n \sqrt{\frac{\omega_n}{\varepsilon_0}}\, f_n(t)\, \vec{e}_n(\bar{r}) & (24a) \\ B(\vec{r},t) = \sum\limits_n \sqrt{\mu_0 \omega_n}\, g_n(t)\, \vec{b}_n(\vec{r}) & (24b) \\ j(\vec{r},t) = \sum\limits_n \sqrt{\varepsilon_0 \omega_n}\, j_n(t)\, \vec{e}_n(\vec{r}) & (24c) \end{cases}$$

(transverse current)

Remark:

- Often normalization factor $\gamma = \omega_n$ is chosen
- Representation (24) is a summary form for discrete infinite system of modes, continuos system of modes is achieved by the limit of infinite boundaries, $V \to \infty$, in the final answer.

29.04.09

- Here $\{\vec{e}_n(\vec{r})\}$ is basis. for travsvers vector fields

$$\Rightarrow \quad \vec{e}_n^{\,*}(r) = \sum_{n'} u_{nn'} \vec{e}_{n'}(r) \qquad (25a)$$

$$\vec{e}_n(r) = \sum_{n'} u_{nn*}^{*} \vec{e}_{n'}^{\,*}(r) = \sum_{\substack{n'\\ n''}} u_{nn'}^{*} u_{n'n''} \vec{e}_{n''}(r) \quad ($$

$$\Rightarrow \quad \sum_{n'} u_{nn'}^{*} u_{n'n''} = \delta_{nn''} \qquad (25c)$$

For real modes $u_{nn'} = \delta_{nn'}$

- Mode equation (18) is real (has real coefficien

with $\vec{e}_n, \vec{b}_n$ solutions $\vec{e}_n^{\,*}$ and $\vec{b}_n^{\,*}$ are alsosolutions with same ω_n when

$$\Rightarrow \quad u_{nn'} = 0 \quad \text{if } \omega_n \neq \omega_{n'} \qquad (25d)$$

- $$\vec{b}_n^{\,*} \underset{(18a)}{=} \frac{c}{\omega_n} \operatorname{rot} \vec{e}_n^{\,*} \underset{(25a)}{=} \frac{c}{\omega_n} \sum_{n'} u_{nn'} \operatorname{rot} \vec{e}_{n'}(\vec{r}) =$$

$$\underset{(18a)}{=} \frac{c}{\omega_n} \sum_{n'} \underbrace{u_{nn'} \frac{\omega_{n'}}{c}}_{(25d)\ = u_{nn'}\frac{\omega_n}{c}} \vec{b}_{n'}(\vec{r}) = \sum_{n'} u_{nn'} b_{n'} \qquad (25c)$$

- $\vec{E}$ is a real field

$$\vec{E} \underset{(24a)}{=} \sum_n \sqrt{\frac{\omega_n}{\varepsilon_0}} f_n \vec{e}_n \underset{E\ \text{real}}{=} \sum_{n'} \sqrt{\frac{\omega_{n'}}{\varepsilon_0}} f_{n'}^{*} \vec{e}_{n'}^{\,*} \underset{(25a)}{=} \sum_{nn'} \sqrt{\frac{\omega_n'}{\varepsilon_0}} \times$$

$$\times\, u_{n'n*} \vec{e}_n f_{n'}^{*} \underset{(25d)}{=} \sum_n \sqrt{\frac{\omega_n}{\varepsilon_0}} e_n \sum_{n'} u_{n'm} f_{n'}^{*}$$

$$\Rightarrow \quad f_n = \sum_{n'} u_{n'n} f_{n'}^{*} \qquad (26a)$$

$$f_n^{*} = \sum_{n'} u_{n'n}^{*} f_{n'} \qquad (26b)$$

Similarly $\quad g_n = \sum_{n'} u_{n'n} g_{n'}^{*} \quad (26c) \qquad g_n^{*} = \sum_{n'} u_{n'n}^{*} g_{n'} \quad (26d)$

$\quad j_n = \sum_{n} u_{n'n} j_{n}^{*} \; \cancel{e} \,(26e) \qquad j_n^{*} = \sum_{n'} u_{n'n}^{*} j_{n'} \quad (26f)$

~~An$a~~tz (24) substituted in ME gives

$$\mathrm{rot}\,\vec{E} \underset{(24a)}{=} \sum \sqrt{\frac{\omega_n}{\varepsilon_0}} f_n\, \mathrm{rot}\,\vec{e}_n \underset{(18a)}{=} \sum_n \sqrt{\frac{\omega_n}{\varepsilon_0}} f_n \frac{\omega_n}{c} \vec{b}_n =$$

$$\underset{(10a)}{=} -\dot{\vec{B}} \underset{(24b)}{=} -\sum_n \sqrt{\mu_0 \omega_n}\, \dot{g}_n b_n$$

$$\Rightarrow \qquad \dot{g}_n = -\omega_n f_n \qquad (27a)$$

Similarly from Eq. (10b) and (24)

$$\dot{f}_n = \omega_n g_n - j_n \qquad (27b)$$

Equation (27) can be written in a different form by using

$$\alpha_n = f_n + i g_n \qquad (28a)$$

$$\beta_n = f_n - i g_n \qquad (28b)$$

~~As~~ which has an inverse

$$f_n = \frac{1}{2}(\alpha_n + \beta_n) \qquad (28c)$$

$$g_n = \frac{1}{2i}(\alpha_n - \beta_n) \qquad (28d)$$

$$\Rightarrow \quad \dot{\alpha}_n \underset{(28a)}{=} \dot{f}_n + i\dot{g}_n \underset{(27)}{=} \omega_n g_n - j_n - i\omega_n f_n =$$

$$= -i\omega_n (f_n + i g_n) - j_n$$

$$\underset{(28a)}{=} -i\omega_n \alpha_n - \boxed{f_n}\, j_n \qquad (29a)$$

Similarly

$$\Rightarrow \qquad \beta_n = i\omega_n \beta_n - j_n \qquad (29b)$$

From Eqs. (28) and (29) follows

$$\beta_n \underset{(28b)}{=} (f_n^* + i g_n^*)^* \underset{(26b,d)}{=} \Big[\sum_{n'} u_{n'n}^* \underbrace{(f_{n'} + i g_{n'})}_{\alpha_{n'}}\Big]^* = \sum_{n'} u_{n'n} \alpha_{n'}^* \qquad (3$$

$\alpha_n(t)$ are <u>normal coordinates</u>, they fully describe the fields $\vec{E}(\vec{r},t)$; $\vec{B}(\vec{r},t)$ for all $\vec{r}$ and t.

=> Real canonical coordinates for the modes are defined as

$$q_n = \frac{1}{2}(\alpha_n + \alpha_n^*) \qquad (31a)$$

$$p_n = \frac{1}{2i}(\alpha_n - \alpha_n^*) \qquad (31b)$$

Substituting (31) in Eq.(29) gives

$$\dot{q}_n = \omega_n p_n - \mathrm{Re}[j_n] \qquad (32a)$$

$$\dot{p}_n = -\omega_n q_n - \mathrm{Im}[j_n] \qquad (32b)$$

Eq.(32) are fully equivalent to ME (10). Eq.(32) are equivalent to ~~the~~ Hamilton equations with the canonical Hamiltonian

$$H = \sum_n \left\{ \frac{\omega_n}{2}(p_n^2 + q_n^2) - p_n \mathrm{Re}[j_n] + q_n \mathrm{Im}[j_n] \right\} \qquad (33)$$

Hamiltonian equations

$$\dot{q}_n = \frac{\partial}{\partial p_n} H = \omega_n p_n - \mathrm{Re}[j_n]$$

$$\dot{p}_n = -\frac{\partial}{\partial q_n} H = -\omega_n q_n - \mathrm{Im}[j_n]$$

Mistake Eq. (22)

$$\vec{e}_{k,+} = (e_{k,1} + i e_{k,2}) \frac{1}{\sqrt{2}} \qquad (22a)$$

$$\vec{e}_{k,-} = (i e_{k,1} + e_{k,2}) \frac{1}{\sqrt{2}} \qquad (22b)$$

Equation (25a) $\vec{e}_n^{\,*} = \sum_{n'} \vec{e}_n u_{nn'}$

$$\vec{e}_n = \vec{e}_{k,+} \cancel{e^{ikr}} = \frac{1}{\sqrt{2}} \underbrace{(\vec{e}_{k,1} + i \vec{e}_{k,2})}_{\text{vector}} \cdot \underbrace{e^{ixr}}_{\text{spacial dependence}}$$

$$\vec{e}^{\,*}_{k,+} = \underbrace{-i}_{u_{nn'}} \vec{e}_{-k,-} \qquad \text{(from Eq. 25)}$$

$$\Rightarrow \quad f^{*}_{k,+} = -i f^{*}_{-k,-} \qquad \text{(from Eq. 26).}$$

Linear polarization $\vec{e}_{k,1}$ $\vec{e}_{k,2}$ real

$$\vec{e}_n = \vec{e}_{k,1} \cdot e^{ikr}$$

$$\vec{e}^{\,*}_n = \vec{e}_{-k,(1,2)}$$

$$f_{k,1} = f^{*}_{-k,1} \qquad f_{k,2} = f^{*}_{-k,2}$$

=> q_n, p_n ~~[illegible]~~ are real canonical conjugate variables.

- A different choice of the variables is possible by using a canonical transformation

However! The factor in the decomposition of the fields (24) changes according to the new representation

III 3. Quantization of the oscillations of the Field

Canonical procedure of quantization
(It is not justified by the classical theory, its correctness is verified by applications)

Substitute "coordinates", q_n, and "momenta" "impulses", p_n, by ~~the~~ operators, $\hat{q}_n$ and $\hat{p}_n$, with the following cannonical commutation relations

$$[\hat{p}_n, \hat{q}_n] = -i\hbar\,\delta_{nm} \quad (34a)$$

$$[\hat{q}_n, \hat{q}_m] = [\hat{P}_n, \hat{P}_m] = 0. \quad (34b)$$

~~[illegible]~~ So, we follow the rule:
Canonical commutator relations for a pair of real canonical conjugate variables, that are independent degrees of freedom.

q_n and p_n represent real degrees of freedom (observable), ~~represent~~ => $\hat{q}_n$ and $\hat{p}_n$ are Hermitian operators.

(Hamiltonian)

- Hamilton~~ian~~ operator is obtained by substitution $q_n \to \hat{q}_n$; $p_n \to \hat{p}_n$ from the classical Hamilton function

$$\hat{H} = \hat{H}_F + \hat{H}_{ext} \quad (35a)$$

Operator for the field energy:

$$\hat{H}_F = \sum_n \frac{\omega_n}{2}\left(\hat{p}_n^2 + \hat{q}_n^2\right) \qquad (35b)$$

Operator of the "~~exchange~~ interaction" energy, that describe interaction between quantum fields and external "classical" variables (current)

$$\hat{H}_{ext} = \sum_n \left\{\hat{q}_n Im[j_n] - \hat{p}_n Re[j_n]\right\} \qquad (35c)$$

- For operators $\hat{q}_n$ and $\hat{p}_n$ and thus for the ~~corresponding~~ derived ~~degrees of freedom, variables~~ quantitis, such as $\hat{H}$, usual rules of Quantum mechanics are valid, in particular
- A classical real observalle quantity O is represent by ~~an~~ operator $\hat{O}$ so that possible ~~measured~~ values of measureble quantities of O are given by eigen values O_ν of $\hat{O}$.
- Information of the state of the system is represented using:
  a) state vector $|\psi\rangle$ in the Hilbertspace, in which $\hat{O}$ acts (pure state)
  b) statistical operator $\hat{\rho}$ (mixed state)
- Average values (over the ensemble of states) can be found as a) $\langle O\rangle = \langle\psi|\hat{O}|\psi\rangle$

12.5 b) $\langle O\rangle = Sp(\hat{\rho}\hat{O}) = Tr(\hat{\rho}\hat{O}$

- Time evolution of the system can be described in different pictures, for example

Schrödinger picture:

- $\hat{O}$ has no time dependence, apart frome explicite parameter dependence $\hat{O}(t)$
→ State $|\psi\rangle$ satisfies Schrödinger Equation

$$i\hbar \partial_t |\psi\rangle = \hat{H} |\psi\rangle \qquad (36)$$

- Statistical operator satisfies Liouville - von Neumann equation

$$i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}] \qquad (37)$$

Heisenberg picture:

- Time dependence of $\hat{O}$ is given by the operator equation

$$i\hbar \partial_t O = [\hat{O}, \hat{H}] + i\hbar \partial_t \Big|_{\text{explicite}} \hat{O}(t) \qquad (38)$$

$$i\hbar \frac{d}{dt} O = [O, H] + i\hbar \frac{\partial}{\partial t} O(t)$$

- $|\psi\rangle$ and $\hat{\rho}$ do not depend on time.

- By measuring eigenstates O_ν the system is prepared in ~~a~~ ~~the~~ a associated state $|O_\nu\rangle$.

II. 4. Heisenberg-~~picture~~ equations for quantum fields oscillations

$$\dot{\hat{q}}_n = \frac{i}{\hbar}[H,\hat{q}_n] \underset{(34,35)}{=} \omega_n \hat{P}_n - \mathrm{Re}[j_n] \quad (39a)$$

$$\dot{\hat{p}}_n = \frac{i}{\hbar}[H,\hat{p}_n] \underset{(34,35)}{=} -\omega_n \hat{q}_n - \mathrm{Im}[j_n] \quad (39b)$$

Eq. (39) is a quantum form of classical Eqs (32)

Combining Eq. (39) with classical transformation (38)

$$\hat{a}_n = \frac{1}{\sqrt{2\hbar}}(\hat{q}_n + i\hat{p}_n) \quad (40a)$$

$$\hat{a}_m^+ = \frac{1}{\sqrt{2\hbar}}(\hat{q}_n - i\hat{p}_n) \quad (40b)$$

Inverse transformation

$$\hat{q}_n = \sqrt{\frac{\hbar}{2}}(\hat{a}_n + \hat{a}_n^+) \quad (40c)$$

$$\hat{p}_n = \frac{1}{i}\sqrt{\hbar/2}(\hat{a}_n^+ - \hat{a}_n^+) \quad (40d)$$

$$\Rightarrow \quad \dot{\hat{a}}_n = \frac{1}{\sqrt{2\hbar}}(\dot{\hat{q}}_n + i\dot{\hat{p}}_n) \underset{(39)}{=} -i\omega_n \hat{a}_n - \frac{j_n}{\sqrt{2\hbar}} \quad (41a)$$

$$\dot{\hat{a}}_n^+ = i\omega_n \hat{a}_n^+ - \frac{j_n^*}{\sqrt{2\hbar}} \quad (41b)$$

Eq. (41) is a quantum version of Eq. (29)

$$[\hat{a}_n, \hat{a}_m^+] = \delta_{nm} \quad (42a)$$

$$[a_n, a_m] = [a_m^+, a_n^+] = 0 \quad (42b)$$

III. 2 Field oscillations ~~in~~ without maters interaction

$$\dot{j}_n = 0$$

$$\dot{\hat{a}}_n = -i\omega_n \hat{a}_n \qquad (43)$$

$$\Rightarrow \quad \hat{a}_n(t) = e^{-i\omega_n t} a_n(t=0) \qquad (44a)$$

$$\hat{a}_n^+(t) = e^{i\omega_n t} a_n^+(t=0) \qquad (44b)$$

$$\Rightarrow \quad [\hat{a}_n(t), \hat{a}_n(t')] = [\hat{a}_n^+(t), \hat{a}_n^+(t')] = 0. \qquad (45)$$

for all t and t'. For real observables one obtain

$$\hat{q}_n(t) \underset{(40c,44)}{=} \sqrt{\hbar/2}\left(\hat{a}_n(0)e^{-i\omega_n t} + \hat{a}_n^+(0)e^{i\omega_n t}\right) =$$

$$\underset{(40a,40b)}{=} \frac{1}{2}\left(\left[\hat{q}_n(0) + i\hat{p}_n(0)\right]e^{-i\omega_n t} + \left(\hat{q}_n(0) - i\hat{p}_n(0)\right)e^{i\omega_n t}\right) =$$

$$= \hat{q}_n(0)\cos\omega_n t + p_n(0)\sin\omega_n t \qquad (46a)$$

Analogously:

$$\hat{p}_n(t) = \hat{p}_n \cos\omega_n t - \hat{q}_n(0)\sin\omega t \qquad (46b)$$

$$\rightarrow \quad [\hat{q}_n(t), \hat{q}_n(t')] \underset{(46a)(34)}{=} \frac{\hbar}{i}\left(\sin\omega_n t\cos\omega_n t' - \cos\omega_n t\sin\omega_n t'\right.$$

$$= \frac{\hbar}{i}\sin(\omega_n(t-t')) \qquad (47)$$

$\Rightarrow$ $\hat{q}_n(t)$ commutes with $\hat{q}_n(t')$ only at times $\omega_n(t-t') = l\pi$, l is integer $(0, \pm 1, \pm 2 \ldots)$ (48)

- when Eq. (48) is satisfied $\hat{q}(t)$ and $\hat{q}(t')$ have the same set of eigenstates.

=> A System can be prepared such that ~~in both~~ ~~[illegible]~~ the fluctuation ~~in~~ between measurements in times t and t' is arbitrary small. Once the 2 measurement is done at t' such that Eq.(48) is not fulfilled, the f ?

- Analogous is true for $\hat{p}_n(t)$ and $\bar{p}_n(t')$
- Eq.(45) means the possibility of the fluctuation les dynamics for "Amplitude" operators a and a^+

However: a and a^+ are not Hermitian operators and ~~are~~ thus do not represent measuble quantities

In particular one cannot prepare the system via measurement of $\hat{a}$ gantities in an associated eigenstate.

III.§3 Field operators

With $q_n, p_n \rightarrow \hat{q}_n, \hat{p}_n$ one obtains $\vec{E}, \vec{B} \rightarrow \hat{\vec{E}}, \hat{\vec{B}}$

Explicit form is obtained from Eqs. (28) and (2

$$f_n \underset{(28c)}{=} \frac{1}{2}(\alpha_n + \beta_n) \underset{(30)}{=} \frac{1}{2}\Big(\alpha_n + \sum_{n'} u_{nn'}\alpha_{n'}^*\Big) \qquad (49a)$$

$$g_n \underset{(28d)}{=} \frac{1}{2i}(\alpha_n - \beta_n) \underset{(30)}{=} \frac{1}{2i}\Big(\alpha_n - \sum_{n'} u_{nn'}\alpha_{n'}^*\Big) \qquad (49b)$$

$$\Rightarrow \vec{E} \underset{(24a)}{=} \sum_n \sqrt{\frac{\omega_n}{\varepsilon_0}}\, f_n \cdot \vec{e}_n \underset{(49a)}{=} \underbrace{\frac{1}{2}\sum_n \sqrt{\frac{\omega_n}{\varepsilon_0}}\,\alpha_n \vec{e}_n}_{\vec{E}^{(+)}} + \underbrace{\frac{1}{2}\sum_{nn'} \sqrt{\frac{\omega_n}{\varepsilon_0}}\, U_{n'n}\alpha_{n'}^* \vec{e}_n}_{\vec{E}^{(-)}}$$

$$\Rightarrow E^{(-)} \underset{\substack{n\leftrightarrow n' \\ (25d)\ \omega_n = \omega_{n'}}}{=} \frac{1}{2}\sum_n \sqrt{\frac{\omega_n}{\varepsilon_0}}\,\alpha_n^* \underbrace{\sum_{n'} u_{nn'}\vec{e}_{n'}}_{(25a)\ \vec{e}_n^*} = \frac{1}{2}\sum_n \sqrt{\frac{\omega_n}{\varepsilon_0}}\,\alpha_n^* \vec{e}_n^* = E_n^{(+)}$$

$$\Rightarrow \vec{E} = \sum_n \frac{1}{2}\sqrt{\frac{\omega_n}{\varepsilon_0}}\,\alpha_n^{(+)}\vec{e}_n(r) + c.c. \underset{(31)}{=} \sum_n \frac{1}{2}\sqrt{\frac{\omega_n}{\varepsilon_0}}\,(q_n + i p_n)\vec{e}_n(r) + c.c. \quad (5$$

From Eq. (28) and (30)

~~$f_n \underset{(28c)}{=} \frac{1}{2}(\alpha_n + \beta_n) = \frac{1}{2}\left(\alpha_n + \sum_{n'} u_{n'n}\alpha^*_{n'}\right)$ (49a)~~

~~$g_n \underset{(28d)}{=} \frac{1}{2i}(\alpha_n - \beta_n) \underset{(30)}{=} \frac{1}{2i}\left(\alpha_n - \sum'_{n'} u_{n'n}\alpha^*_{n'}\right)$ (49b)~~

~~$\Rightarrow \vec{E} \underset{(24a)}{=} \sum_n \sqrt{\frac{\omega_n}{\varepsilon_0}} f_n \vec{e}_n \underset{(49a)}{=} \frac{1}{2}\sum_n \sqrt{\frac{\omega_n}{\varepsilon_0}}\,\alpha$~~

Substitituting $q_n, p_n \rightarrow \hat{q}_n, \hat{p}_n$ into expession for $\vec{E}$

$$\hat{E} = \hat{E}^{(+)} + \hat{E}^{(-)} \qquad (51a)$$

$$\hat{E}^{(+)} = \sum_n \frac{1}{2}\sqrt{\frac{\omega_n}{\varepsilon_0}}\,(\hat{q}_n + i\hat{p}_n)\,\vec{e}_n(\vec{r}) \underset{(40a)}{=} \sum_n \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}}\,\hat{a}_n \vec{e}_n(\vec{r}) \quad (51$$

$$\hat{E}^{(-)} = \left(\hat{E}^{(+)}\right)^{+} = \sum_n \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}}\,\hat{a}_n^{+}\vec{e}_n^{\,*}(\vec{r}) \qquad (51c)$$

Analogously one obtains

$$\hat{\vec{B}}^{*} = \hat{\vec{B}}^{(+)} + \hat{\vec{B}}^{(-)} \qquad (52a)$$

$$\hat{\vec{B}}^{(+)} = \sum_n \frac{1}{2i}\sqrt{\mu_0\omega_n}\,(\hat{q}_n + i\hat{p}_n)\,\vec{b}_n(\vec{r}) = \sum_n \sqrt{\frac{\mu\hbar\omega_n}{2}}\,\frac{\hat{a}_n}{i}\vec{b}_n \quad (52b$$

$$\hat{\vec{B}}^{(-)} = \left(\hat{\vec{B}}^{(+)}\right)^{+} = -\sum_n \sqrt{\frac{\mu_0\omega_n\hbar}{2}}\,\frac{a_n^{+}}{i}\vec{b}_n(\vec{r}) \qquad (52c)$$

- Vector character and $\vec{r}$ dependence of the field operators are in $\vec{e}_n(\vec{r})$ and $\vec{b}_n(\vec{r})$.
- Operator character follows from $\hat{a}_n, \hat{a}_n^{+}$ or $\hat{q}_n$ and $\hat{p}_n$
- Depending on the choice of the representation (Schrödinger or Heisenberg) $\hat{q}_n, \hat{p}_n, \hat{a}_n, \hat{a}_n^{+}$ and $\vec{E}$ and $\vec{B}$ depend on time t or not.
- Operators $\hat{\vec{E}}^{(+)}, \hat{\vec{B}}^{(+)}$ $(\hat{\vec{E}}^{(-)}, \hat{\vec{B}}^{(-)})$ contain only Opepators $\hat{a}_n^{*}$ $(\hat{a}_n^{+})$

14.05.09

- Operators ~~$E^{(+)}$ can be exchanged, $\hat{E}^{(+)}, \hat{B}^{(+)}$~~ ~~$\hat{\vec{E}}^{(+)}$ and $\hat{\vec{B}}^{(+)}$ $(\hat{\vec{E}}^{(-)}, \hat{\vec{B}}^{(-)})$ commute, $(+)$ do not commute with $(-)$.~~
- ~~$\vec{E}$ do not commute with $\vec{B}$~~

~~$\Rightarrow$ $\hat{E}^{(+)}$ and $\hat{B}^{(+)}$ commute, $[\hat{E}^{(+)}, \hat{B}^{(+)}] = 0$~~

~~$\hat{E}^{(-)}$ and $\hat{B}^{(-)}$ also commute~~

~~while $\hat{B}, \hat{E}^{(+)}$ and $\hat{B}, \hat{E}^{(-)}$ do not commute~~

~~$[\hat{\vec{B}}^{(+)}, \hat{\vec{B}}^{(-)}] \neq 0.$~~

~~$\Rightarrow$ $\hat{\vec{E}}$ and $\vec{B}$ do not commute, consequently one cannot measure $\vec{E}$ and $\vec{B}$ simultaneously.~~

- Recalling $\hat{a}_n(t) = e^{-i\omega_n t}\hat{a}_n(0)$; $\hat{a}_n^+ = e^{i\omega_n t}\hat{a}_n^+(0)$ one obtains that (see Eq. (51), (52)) $\hat{\vec{E}}^{(+)}$ and $\hat{\vec{B}}^{(+)}$ contains only $e^{-i\omega_n t}$ while $\vec{E}^{(-)}, \vec{B}^{(-)}$ has only $e^{+i\omega_n t}$ terms. ~~One says that~~ $\hat{\vec{E}}^{(+)}$ and $\hat{\vec{B}}^{(+)}$ are positive frequency components, while $\vec{E}^{(-)}, \hat{\vec{B}}^{(-)}$ are negative frequency components.

II. ~~6~~ 3 Commutators of field operators

II. ~~6~~ 3.1 Commutators ~~[illegible]~~ at the same time (local)

Using Eqs. (51), (52), (42B) we obtain

$$[\hat{\vec{E}}^{(\pm)}_{\ell}(r), \hat{\vec{B}}^{(\pm)}_{\ell'}(r')] = [\hat{\vec{E}}_{\ell}^{(\pm)}(r), \hat{\vec{E}}^{(\pm)}_{\ell'}] = [\hat{\vec{B}}^{(\pm)}_{\ell}(r), \hat{\vec{B}}^{(\pm)}_{\ell'}(r')] = 0 \qquad (53)$$

$$[\hat{\vec{E}}^{(-)}_{\ell}(r), \hat{\vec{B}}^{(+)}_{\ell'}(r')] \underset{(51c,52B)}{=} \sum_{n,n'} \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}\,\frac{\mu_0\hbar\omega_n}{2}}\;\frac{1}{i}\left(\vec{e}_n^{\,*}(\vec{r})\right)_{\ell}\left(\vec{b}_{n'}(r')\right)_{\ell'} \times \underbrace{[\hat{a}_n^+, \hat{a}_{n'}]}_{(42)\,=\,-\delta_{nn'}} = \frac{i\hbar}{2\varepsilon_0}\sum_n \frac{\omega_n}{c}\left(\vec{e}_n^{\,*}(r)\right)_{\ell}\left(\vec{b}_n(r')\right)_{\ell'} =$$

$$= (\text{will be shown}) = -\frac{i\hbar}{2\varepsilon_0}\sum_j \varepsilon_{j\ell\ell'}\frac{\partial}{\partial x_j}\delta(\vec{r}-\vec{r}\,') \qquad (54)$$

where $\varepsilon_{j\ell\ell} = \begin{cases} 1 & \text{when } (j,\ell,\ell') \text{ is even perm. of } (1,2,3) \\ -1 & \text{when } -\text{''}- \text{ is uneven perm.} -\text{''}- \\ 0 & \text{when one index is equal others} \end{cases}$

Levi-Civita-Tensor fully antysymmetric

$$\int dx\, f(x)\frac{\partial}{\partial x}\delta(x-x') = -\int dx\,\frac{\partial f}{\partial x}\delta(x-x') = -f'\big|_{x=x'} \qquad (56)$$

Similarly:

$$\left[\left(\hat{\vec{E}}^{+}(r)\right)_{e}, \hat{\vec{B}}^{-}_{e'}(r')\right] = \left[\hat{\vec{E}}^{(-)}_{e}(r), \vec{B}^{(+)}_{e'}(r')\right] \qquad (57)$$

thus ist

$$\left[\hat{E}_{e}(r)_{\cancel{e}}, \hat{\vec{B}}_{e'}(r')\right] \underset{(51,52,53)}{=} \left[\hat{\vec{E}}^{(+)}_{e}(r), \hat{\vec{B}}^{(-)}_{e'}(r')\right] + \left[\hat{E}^{(-)}_{e}(r), \hat{\vec{B}}^{(+)}_{e'}(r')\right]$$

$$\underset{(54,57)}{=} \frac{\hbar}{i\varepsilon_0} \sum_{j} \varepsilon_{jee'} \frac{\partial}{\partial x_j} \delta(x-x') \qquad (58)$$

=> • Thus at finite distance $|r-r'| > 0$, $\hat{\vec{E}}$ and $\hat{\vec{B}}$ commute and thus can be measured simultaneously.

• Because of Eqs. (55) and (58) same components of $\hat{E}$ and $\hat{B}$ commute at all $\vec{r}$ and $\vec{r}\,'$ and thus can be measured simultaneously.

Similarly: $\left[\hat{\vec{E}}_{e}(\vec{r}), \hat{\vec{E}}_{e'}(\vec{r}\,')\right] = \left[\hat{\vec{B}}_{e}(\vec{r}), \hat{\vec{B}}_{e'}(\vec{r}\,')\right] = 0$ (59

~> The fields E and B can be measured simultaneouly at any ~~s~~ space-points.

Derivation of Eq. (54)

- Let $\vec{W}(r)$ is a rotational vector and $\vec{Q}(r)$ is a sourese vector, than

$$\int d^3r\, \vec{W}^*(\vec{r}) \cdot Q(\vec{r}) = 0 \qquad (60)$$

Eq. (60) can be proved by considering fourier decomposition of the fields $\vec{W}$ and $\vec{Q}$ and using orthogonal longitudinal and transverse components

- Since $\vec{e}_n$ is a complete orthonormal basis one can use it to represent any vector $\vec{V}_W$ $(\vec{V} = \vec{V}_W + \vec{V}_Q)$.

$$\left(\vec{V}_W(r')\right)_{e'} = \sum_n \left(\vec{e}_n(r')\right)_{e'} \cdot \int d^3r \left(\vec{e}_n^{\,*}(\vec{r}) \cdot \vec{V}_W(r)\right) = \qquad [\vec{V}_W = V - \vec{V}_Q]$$

$$= \sum_n \vec{e}_n(r')_{e'} \cdot \int dr^3 \left(\vec{e}_n^{\,*}(r) \cdot \vec{V}(r)\right) =$$

$$= \sum_e \int dr^3\, \delta^{\perp}_{ee'}(r', r)\, \vec{V}(r)_e \qquad (61)$$

Transverse δ-function $\quad \delta^{\perp}_{ee'}(r, r') = \sum_n \vec{e}_n(r')_{e'} \cdot \vec{e}_n^{\,*}(r)_e$

projectes a vector field onto its transverse component

$$\Rightarrow \sum_{je} \varepsilon_{jee'} \frac{\partial}{\partial x_j} \vec{V}(r')_e = \mathrm{rot}\,[\vec{V}(r')]_{e'} \underset{(2)}{=} \left(\mathrm{rot}\, \vec{V}_W(r')\right)_{e'} =$$

$$\underset{(61)}{=} \sum_n [\mathrm{rot}\, \vec{e}_n(r')]_{e'} \int d^3r \left(\vec{e}_n^{\,*}(r) \cdot \vec{V}(r)\right) \underset{(18)}{=}$$

$$\underset{(18)}{=} \sum_e \int d^3r \left\{ \sum_n \frac{\omega_n}{c} \vec{b}_n(r')_{e'} \cdot \vec{e}_n^{\,*}(r)_e \right\} \vec{V}(\vec{r})_e \qquad (62)$$

Comparing Eq. (56) with Eq. (62) shows ($\vec{V}(r)$ is a test field)

$$\sum_n \frac{\omega_n}{c} \vec{e}_n^{\,*}(\vec{r})_e\, \vec{b}_n(\vec{r}')_{e'} = +\sum_j \varepsilon_{jee'} \frac{\partial}{\partial x_j} \delta(\vec{x} - \vec{r}') \qquad (63)$$

This proves the relation in Eq. (64).

III. ~~6.2~~ 3.2 Commutators of the fields at different times t.

- Commutators at the same time instants are calculated from Eq. (42) for operators $\hat{a}_n^+, \hat{a}_n$ without accounting for equation of motion for operators
- Computation of the commutators at different times require solution of Heisenberg equation for operators for free field operators, as in Eq. (44).
- Due to Eq. (47) it is expected, that at different times the field operators ~~are~~ do not commute.

Taking the decomposition of fields into the modes ~~ad~~ and using Eq (44) one obtains (without derivation)

$$\left[\hat{E}(r,t)_e, \hat{\bar{E}}(r',t')_{e'}\right] = \frac{i\hbar c}{\varepsilon_0}\left(\frac{\delta_{ee'}}{c^2}\frac{\partial^2}{\partial t \partial t'} - \frac{\partial^2}{\partial x_e \partial x_{e'}}\right) D(r-r', t-t') \quad (64)$$

with $D(r-r', t-t') = \frac{1}{4\pi|\bar{r}|}\Big(\delta(|r|+ct) - \delta(|r|-ct)\Big)$ (65)

- The fields do not commute only at the poins in the time-space ~~dt~~ which a light sygnal can connect

$|r-r'| = ct$
~~$|r-r'| = ct$~~.

- Similarly, the commutators between $\hat{E}$ and $\hat{B}$, and $\hat{B}$ and $\hat{B}$ can be obtained.

$\pm ct = r$ — evidence of delay in interaction

IV. Quantum states of radiation fields

IV. 1 Eigenstates of energy operator (with definite number of phonons)

From Eq. (35b) and (40) follows for the energy operator

$$\hat{H}_F = \sum_n{}' \hat{H}_{F_n} \qquad (66a) \quad (61a)$$

$$\hat{H}_{F_n} = \frac{\omega_n}{2}\left(\hat{p}_n^2 + \hat{q}_n^2\right) \underset{(40)}{=} \frac{\hbar\omega_n}{4}\Big(-(\hat{a}_n - \hat{a}_n^+)(\hat{a}_n - \hat{a}_n^+) +$$

$$+ (\hat{a}_n + \hat{a}_n^+)(\hat{a}_n + \hat{a}_n^+)\Big) = \frac{\hbar\omega_n}{4}\left(\hat{a}_n\hat{a}_n^+ + \hat{a}_n^+\hat{a}_n + \hat{a}_n^+\hat{a}_n + \hat{a}_n\hat{a}_n\right.$$

$$\underset{(42a)}{=} \hbar\omega_n\left(\hat{a}_n^+\hat{a}_n + \frac{1}{2}\right) \qquad (66b) \quad (61b)$$

- $\hat{H}_F$ is energy operator of a system of uncoupled harmonical oscillators.

⇒ Quantum states of the free field is a direct (tensor) product of states of single oscillator

$$R_F = R_{F_1} \otimes R_{F_2} \otimes R_{F_3} \cdots \otimes R_{F_n} \otimes \cdots \qquad (62) \quad (67)$$

[two particles $\psi_1(x_1)$, $\psi_2(x_2) \Rightarrow \psi = \psi_1(x_1)\psi_2($

- Construction of a state R_{F_n} for a single oscillator is described in a QM course.

⇒ Result: there is one state withe the lowest energy (ground states, vacuum state) $|0_n\rangle$ such that $H_{F_n}|0_n\rangle = \frac{1}{2}|0_n\rangle$ (68) (63)

- Othe possible eigenvalues of H_F are

$$\mathcal{E}_{nj_n} = \hbar\omega_n\left(j_n+\frac{1}{2}\right), \quad j_n = 0, 1, 2 \ldots \qquad (69) \; (64)$$

- For excited states

$$\hat{a}_n^+ |j_n\rangle = \sqrt{j_n+1}\, |j_n+1\rangle \qquad (70a) \; 65a$$

$$\hat{a}_n |j_n\rangle = \sqrt{j_n}\, |j_n-1\rangle \qquad (70b) \; 65b$$

$\Rightarrow$ $\hat{a}_n^+, \hat{a}_n$ are "rising" and "lowering" operators, or "creation" and "annyhilation" operators for mode "n".

- The state $|j_n\rangle$ can be obtained from the vacuum state $|0_n\rangle$ as

$$|j_n\rangle = \frac{(\hat{a}_n^+)^{j_n}}{\sqrt{j_n!}}\, |0_n\rangle \qquad (71) \; 66$$

$$\langle j_n | j_n' \rangle = \delta_{j_n, j_n'} \qquad (72) \; 67$$

- Total ground state of the field (vacuum state)

$$|0\rangle = |0_1\rangle \otimes |0_2\rangle \otimes \ldots \otimes |0_n\rangle \otimes \ldots \qquad (73) \; 68$$

- The state with definite energy

$$|\{j_n\}\rangle \equiv |j_1, j_2 \ldots j_n \ldots\rangle = \prod_n \frac{(\hat{a}_n^+)^{j_n}}{\sqrt{j_n!}}\, |0\rangle \qquad (74 \; 69$$

Fock-State

- The states $|\{j_n\}\rangle$ create a basis in a space of states: an arbitrary state of fields can be constructed from as a combination of $|\{j_n\}\rangle$

- это дает (This gives)

$$\hat{N}_n |j_n\rangle = j_n |j_n\rangle \qquad (75a) \quad 70a$$

where $$\hat{N}_n = \hat{a}_n^+ \hat{a}_n \qquad (75b) \quad 70b$$

=> $\hat{N}_n$ counts the number of "quants" of energy of the mode. Such quantums of ~~light~~ ligt have characteristics (properties) of particles and will be called photons.

-> The total number of photons is given by

$$\hat{N} = \sum_n{}' \hat{N}_n \qquad (76) \quad 71$$

- Neither energy operator, ~~nor~~ $\hat{H}$, nor number of photons operator, $\hat{N}$, commute with ~~the field~~ opera-tors $\hat{a}_n^+$ and $\hat{a}_n$, and consequently with $\hat{\vec{E}}$ and $\hat{\vec{B}}$ field operators [Eq(51) - (52)].

-For the Fock space one has:

- the number of phonons given by j_n, does not fluctuate
- Expected values of the field components are $\langle\{j_n\}|\hat{\vec{E}}(r)_e|\{j_n\}\rangle \underset{(51)}{=} \sum_n \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}} \langle\{j_n\}|\hat{a}_n|\{j_n\}\rangle \vec{e}_n(r)_e + \text{c.c.}$

&

$$\underset{(70b)}{=} \sum_n \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}} \langle\{j_n\}| j_1, \dots, j_n - 1, \dots\rangle \sqrt{j_n}\, \vec{e}_n(r)_e + \text{c.c.} = 0$$

(72) (77

(72)

- From general formulas (definition) of fluctuations for an operator $\hat{O}$

$$\langle(\Delta\hat{Q})^2\rangle = \langle(\hat{O} - \langle\hat{O}\rangle)(\hat{O} - \langle\hat{O}\rangle)\rangle \qquad (78) \quad 73$$

For the squared fluctuations of $\vec{E}(r)$ in a Fock Space :

$$< \Delta \hat{E}(r)_e^2 > = \quad < \hat{\vec{E}}(r)_e > =$$

$$\overset{(77)(78)}{\underset{(56)}{=}} \underbrace{<\{j_m\}| \hat{\vec{E}}^{(+)}(r)_e^2 | \{j_m\}> + <\{j_m\}| \vec{E}^{(-)}(r)_e^2 | \{j_m\}>}_{0 \text{ as in Eq. } (77)} =$$

$$+ <\{j_m\}| \left[\hat{\vec{E}}^{(+)}(r)_e \hat{\vec{E}}^{(-)}(r)_e + \hat{\vec{E}}^{(-)}(r)_e \hat{\vec{E}}^{(+)}(r)_e \right] | \{j_m\}> =$$

$$\underset{(51)(70)(72)}{=} \sum_n \frac{\hbar \omega_n}{\varepsilon_0} (j_m + \tfrac{1}{2}) \left| \vec{e}_n(r)_e \right|^2 \qquad (79) \quad 74$$

Similarly for magnetic component

$$<\{j_m\}| \vec{B}(r)_e | \{j_m\}> = 0 \qquad (80) \quad 75$$

$$<\{j_m\}| \Delta \hat{\vec{B}}(r)_e^2 | \{j_m\}> = \sum_n \mu_0 \hbar \omega_n (j_n + \tfrac{1}{2}) \left| \vec{b}_n(r)_e \right|^2 \quad (81) \quad 76$$

- Remarks on particle-like properties of photons.

- Phonons are not localized in space; they are not point-particles. Spatial dependence of phonons is given by the spatial dependence of the modes $\vec{e}(r), \vec{b}(r)$
- The field, constructed only using the transverse modes, cannot form particle-like states, localized in space, $\vec{E} \sim \delta(\vec{r} - \vec{r}_0)$.
- Interpretation of the spatial dependence of the field operators is quite different from that for the classical fields.

Example: Classical field $E(r)$ is characterised by its frequency, ω, and after some normalization to the volume has a total energy, $\hbar\omega$.

If one measures energy of such field ~~in~~ by measuring intensity, ~~the result will~~ in some small volume, this will give energy that is only a small fraction of $\hbar\omega$.

$I \sim |\vec{E}(r)|^2$

In contrast ~~for~~ consider the quantum field, constructed for example using one photon states as

$$|\psi\rangle = \sum_{\substack{n \\ \text{such that } \omega_n=\omega}} c_n \hat{a}_n^{+} |0\rangle \qquad (82) \quad 77$$

which has the same energy $\hbar\omega$. We shall measure the intensity of the field using absorption experiment (for example, utilising, photoelectric effects $\hbar\omega \to \vec{j}$). If the photon is detected in such measurement, it will be of the entire energy $\hbar\omega$, not part of it. The measuring process localizes a photons. in a single point. (it looks like it)

=> The spatial dependence for $\vec{E}(r)$ in ~~the~~ quantum theory does not mean ~~as~~ a continuum distribution of energy in the space, as in the classical theory. Rather it means a probability distribution as in quantum ~~theory~~ mechanics: it yields the probability of the experimental results. at point $\vec{r}$.

- Thus photons are not defined by position or position distribution in the classical sence; they are defined by the probability of the field amplitudes.

=> Photons are not individual particles, each having its on history from creation to destruction.

- Photons are introduced here as ~~quanta~~ Energy Quanta of the electromagneticfields. Other (quantum) particle-like properties ~~are~~ of photons depend on the choice of the basis mode system.

Example: Field momentum

=> Classical expression for the total momentum of the field is

$$\vec{P} = \varepsilon_0 \int d^3r \left[\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right] . \qquad (83)$$ 76

=> Quantised expression

$$\hat{\vec{P}} = \frac{\varepsilon_0}{2}\int d^3r \left\{\left[\hat{\vec{E}}(r,t)\times\hat{\vec{B}}(r,t)\right] - \left[\hat{\vec{B}}(r,t)\times\hat{\vec{E}}(r,t)\right]\right\} \qquad (84)$$ 77

symmetrical form is because $[\hat{\vec{E}},\hat{\vec{B}}] \neq 0$.

=> Decomposition of $\hat{\vec{E}}$ and $\hat{\vec{B}}$ into the modes example in [Eq.(15)].

$$\hat{\vec{P}} = \sum_{k,\lambda} \hbar\vec{k}\,\hat{N}_{k,\lambda} \qquad \hat{N}_{k,\lambda} = \hat{a}^+_{k,\lambda} a_{k\lambda} \qquad (85)$$ 78

=> $j_{k,\lambda}$ in the Fock space now carriers a contribution to the field momentum as $\hbar\vec{k}j_{k,\lambda}$

=> Fock space made of plane waves: eigenstates have definite momentum $\vec{P}$ ($\vec{P}$ is a "good" quantum number.
Fock space made of other states ~~case~~ might carry no momentum, for examle ~~a~~ system of standing waves have total momentum "0".

=> It is possible to construct Fock space where angular momentum is a good quantum number.

III. 2. Vacuum characteristics (properties)

Vacuum state $|0\rangle$ is a special state of the Fock-State with all $j_n = 0$. $|0\rangle$ is the ground state of the field, that does not contain photons.

III. 2.1 Zero energy

Taking Eq. (66) and (69) one obtains

$$E_{vac} = \sum_n \frac{\hbar\omega_n}{2} \qquad (86)$$

- E_{vac} is divergent, because the number of modes is not restricted (is infinite) and $\omega_n \neq 0$.
  Example: for plane waves $\omega_{\lambda,k} = c|k|$
- As long as the mode system is not changed, E_{vac} is infinite but is "constant infinite".
  In this case E_{vac} is not an observable quantity.
  The observable quantity is

$$\hat H_F = \sum_n \hat H_{F_n} = \cancel{\left[\sum_n \hbar\omega_n j_n\right]} = \sum_n \hbar\omega_n \hat N_n \qquad (87a)$$

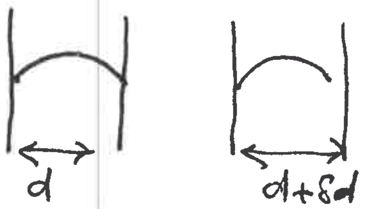
$$\hat H_{\bar F} = \hat H_F - E_{vac} \qquad (87b)$$

- Physically relevant quantity related to E_{vac} appear when the mode system changes, leading to changes in the spectrum of the system. Then $\Delta E_{vac} = E'_{vac} - E_{vac}$ is a measurable quantity (E'_{vac} – vacuum energy with the different mode

=> The calculation of ΔE_{vac} are done using the so-called renormalization methods.

Example: Casimir effect

Consider two metallic plates in vacuum (metal is impenetrable to electromagnetic fields). If the distance between the plates changes, the mode system changes, and thus the vacuum energy also changes.



$$\delta E_{vac} = E_{vac}(d+\delta d) - E_{vac}(d)$$

This change is finite, even though $E_{vac}(d) = \infty$

Change in energy ~~mou~~ when d changes must be compensated by external work, so the force must be applied to the system to change d. Renormalization calculations

1. One chooses ~~the~~ restriction to the summation for E_{vac} the so-called "cut-off" E_c, so that $E_{vac}(d, E_c)$ = finite and $E_{vac}(d, \infty) = \infty$
2. The difference $\delta E_{vac}(E_c) = E_{vac}(E_c, d+\delta d) - E_{vac}(E_c, d)$ is finite in the limit $E_c \to \infty$, i.e. $\delta E_{vac}(\infty)$ = finite

Calculation for casimir effect:

$$E_{vac} = \frac{\hbar}{2}\sum_{n k} \omega_n \qquad \omega_n = c\sqrt{k_x^2 + k_y^2 + \frac{\pi^2 n^2}{d^2}} \qquad E_{vac} = S\hbar \int \frac{dk_x\, dk_y}{(2\pi)^2} \cdot \sum_{n=1}^{+\infty}{}' \omega_n \to \infty$$

"Cut-off" procedure: one choice $E_{vac}^{(s)} = \frac{\hbar}{2}\sum_n{}' \omega_n e^{-s\omega_n} \to$ finite

other possible choice $E_{vac}^{(s)} = \frac{\hbar}{2}\sum_n \omega_n e^{-s^2\omega_n^2}$

For analytical calculations $E_{vac}(s) = \frac{\hbar}{2}\sum_n{}' \omega_n \omega_n^{-s}$ $s > 3$

$$E_{vac}(s,d) = \overset{A}{\cancel{S}}\cdot\hbar \int \frac{dk_x\, dk_y}{(2\pi)^2} \sum_{n=1}^{+\infty}{}' c^{1-s}\left(k_x^2 + k_y^2 + \frac{\pi^2 n^2}{d^2}\right)^{\frac{1-s}{2}} = \frac{A\cancel{S}\cdot\hbar}{4\pi}\sum_{n=1}^{+\infty}\int d\mu\, c^{1-s}\left(\mu + \frac{\pi^2 n^2}{d^2}\right)^{\frac{1-s}{2}}$$

$$= -\frac{\cancel{A}\cdot\hbar}{\cancel{4\pi}}\frac{c^{1-s}}{2d^{3-s}}\frac{\pi^{2-s}}{3-s}\underbrace{\sum_{n=1}^{+\infty} n^{3-s}}_{\zeta(s-3)} = -\frac{\cancel{S}\cdot\hbar\cdot c^{1-s}\cdot\pi^{2-s}}{2d^{3-s}(3-s)}\zeta(-3+s)$$

$\zeta(s-3)$ – Rieman zeta function

This expression can be analyticaly continued in ~~comp~~ plane ~~of~~ complex values of s.

The fource per unit ~~square~~ area is

$$f = \frac{F}{A} = -\frac{1}{A}\frac{\partial E_{vac}(s,d)}{\partial d} = -\frac{\hbar c^{1-s}\pi^{2-s}}{2\cancel{(3-s)}}\zeta(s-3)\frac{3}{d^{4-s}}$$

$$\boxed{f(s\to 0) = -\frac{\hbar c\pi^2}{2}\frac{\zeta(-3)}{d^4} = -\frac{\hbar c\pi^2}{240}\frac{1}{d^4}}$$

In real experiments $F \sim 10^{-18}$ newton, force attracti

Conclusion: Vacuum fluctuations lead to measureble fource; purely quantum effect absent in classical fisisue.

III. 2.2. Vaccuum fluctuations.

- According to Eq. (77) $|0\rangle$ is a state with zero expectation value.
- Fluction of the field is given by Eq (79)

$$\langle \Delta\hat{E}(r)_e^2\rangle_{vac} = \langle 0|\,\Delta\hat{E}(r)_e^2\,|0\rangle = \sum_n \frac{\hbar\omega_n}{2\varepsilon_0}|\vec{e}_n(r)_e|^2 \qquad (88)$$

This expression diverges! For example for plane waves Eq. (19)
$|\vec{e}_n(r)|^2 = \frac{1}{V}$ and then Eq. (88) gives

$$\langle \Delta\hat{\vec{E}}_e^2\rangle_{vac} \sim \frac{1}{V}\sum_n \frac{\hbar\omega_n}{2\varepsilon_0} \sim E_{vac} \qquad (89)$$

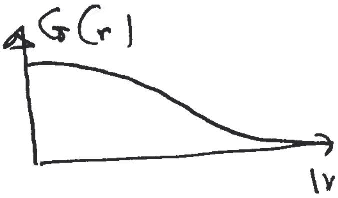
Discussion: Infinite result for fluctuations is not acceptable. Real measuments give finite resuls because:

a) Real detectors have finite spetrum range. Thus real measurements have natural frequency cut-off in Eq. (88)

b) Measumeus are done in finite space, so to model this one has to introduce in Eq. (88) a special function

$$\hat{\vec{E}}(r) = \int d^3r' G(r') \vec{E}(r+r')$$

qualititive appearance of $G(r)$



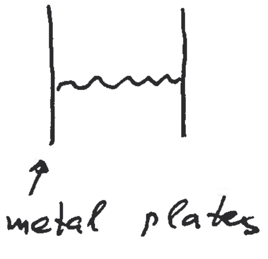
$$\Rightarrow \sum_{e} \langle (\Delta \hat{\vec{E}}(r))_e^2 \rangle_{vac} = \sum_n \frac{\hbar \omega_n}{2\varepsilon_0} \sum_e \left| \int d^3r' G(r') [\vec{e}_n (r+r')]_e \right|^2 \quad (90$$

For plane waves

$\sum_e \left| \int d^3r' G(r') \vec{e}_n(r+r')_e \right|^2 \sim |G(k)|^2$ where $G(k)$ is the Fourier transform of $G(r)$. $G(k) \xrightarrow[k \to \infty]{} 0$

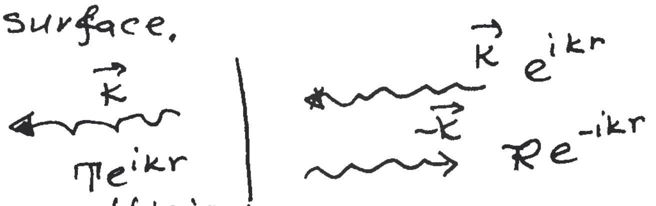
and Eq (90) is convergent.

Example of "cut-off" introduced by real conditions of measument. Casimir effect



- standing electro-magnetic waves inside resonator

Appearance of standing waves is due to reflection of plane waves from metallic surface.



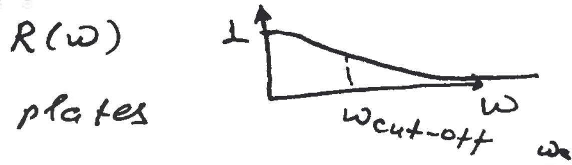
$T(\omega)$ – transmission coefficient

$R(\omega)$ – reflection coefficient

$|T|^2 + |R|^2 = 1$

~~Cassimir effect calculation are based on R(ω) = 1~~

~~Divergency : Divergen~~



When $\omega \gg \omega_{cut\text{-}off}$ – metallic plates do not interact with e-m. waves, thus $E_{vac} = \sum_{n=1}^{\omega_c/\omega_n} \frac{\hbar\omega_n}{2}$

IV. 3. Quasiclassical States (coherent states)

For the quantum states of the field considered so far energy and the amplitude of the fields cannot be measured at the same time.

Question: Is there a system of quantum states of the field which, despite usual quantum mechanical restrictions, can be used to desribe measurable fields $\vec{E}(r)$?

A requirement is that expected values of the energy and the field strength would appoach their classical values in the classical limit $\hbar \to 0$ {Note: zero energy for the states considered before is purely quantum effect, without classical analog).

Parametrising classical states using mode amplitudes α_n [Eq. (28a)]: Classical energy function (Hamiltonian)

$$H_{class}(\{\alpha_n\}) = \sum_n \frac{\omega_n}{2} \alpha_n^* \alpha_n \qquad (91)$$

⇒ The requirement that the correspondig quantum state $|\{\alpha_n\}\rangle$ produced the same energy (with the Hamiltonia operator by Eq (66))

$$\langle\{\alpha_n\}| \hat{H}_F - E_{vac} |\{\alpha_n\}\rangle = \sum_{n'} \hbar\omega_{n'} \langle\{\alpha_n\}| a^+_{n'} a_{n'} |\{\alpha_n\}\rangle$$

$$= \sum_{n'} \cancel{\hbar}\frac{\omega_{n'}}{2} \alpha^*_{n'} \alpha_{n'} \qquad (92)$$

⇒ so one must ensure (must have)

$$\langle\{\alpha_n\}| a^+_{n'} a_{n'} |\{\alpha_n\}\rangle =$$

$$= \frac{1}{2\hbar} \alpha^*_{n'} \alpha_{n'} \qquad (93)$$

Comparing Eqs. (90) and (91) shows, that in order to get the classial valuy of the e-m. field one must have

$$\langle \{\alpha_n\} | a_{n'} | \{\alpha_n\} \rangle = \tilde{\alpha}_{n'} \qquad (94a)$$

$$\tilde{\alpha}_{n'} = \frac{1}{\sqrt{2\hbar}} \alpha_n \qquad (94b)$$

defining $\hat{b}_n = \hat{a}_n - \tilde{\alpha}_n \qquad (95)$

$\Rightarrow$ (94a) $\langle \{\alpha_n\} | \hat{b}_{n'} | \{\alpha_n\} \rangle = 0 \qquad (96)$

$$\Rightarrow \quad \langle \{\alpha_n\} | b^+_{n'} b_{n'} | \{\alpha_n\} \rangle = \langle \{\alpha_n\} | (\hat{a}_{n'} - \tilde{\alpha}_{n'})^+ (\hat{a}_{n'} - \tilde{\alpha}_{n'}) | \{\alpha_n\} \rangle$$

$$= \langle \{\alpha_n\} | \hat{a}^+_{n'} a_{n'} | \{\alpha_n\} \rangle - \alpha^*_{n'} \langle \{\alpha_n\} | \hat{a}_{n'} | \{\alpha_n\} \rangle - \alpha_{n'} \langle \{\alpha_n\} | \hat{a}^+_{n'} | \{\alpha_n\} \rangle$$

$$+ \alpha^*_{n'} \alpha_{n'} = 0 \qquad (97)$$

$\Rightarrow$ The normalization of the state $\hat{b}_{n'} | \{\alpha_n\} \rangle$ is zero.

$$\Rightarrow \quad \hat{b}_{n'} | \{\alpha_n\} \rangle = 0 \qquad (98)$$

$$\underset{(95)}{\Rightarrow} \quad \hat{a}_{n'} | \{\alpha_n\} \rangle = \tilde{\alpha}_{n'} | \{\alpha_n\} \rangle \qquad (99)$$

- Optimal compromise between classical and quantum theories in the sense of agreement between the quantum and classical representations is reached, if the field is represented by eigenstaste of the annihilation operator a_n.
Usually that so-called "Coherent states" are not defined by ~~can~~ the correspondence with the classical fields, but directly from Eq. (99).

Instead of Eq. (99) one can write

$$\hat{a}_{n'}|\{\alpha_n\}\rangle = \alpha_{n'}|\{\alpha_n\}\rangle \qquad (100)$$

This require a slightly different definition of $|\{\alpha_n\}\rangle$

- For "normally" ordered operators (when all $\hat{a}^+$ are on the left of all $\hat{a}$) we have

$$\langle\{\alpha_n\}|\hat{a}^+_{\nu_1}\hat{a}^+_{\nu_2}\dots\hat{a}^+_{\nu_j}\hat{a}_{\mu_1}\dots\hat{a}_{\mu_e}|\{\alpha_n\}\rangle = \\ = \alpha^*_{\nu_1}\alpha^*_{\nu_2}\dots\alpha^*_{\nu_j}\alpha_{\mu_1}\dots\alpha_{\mu_e} \qquad (101)$$

so that for normal ordered operators their expectation Value are obtained by substitution $\hat{a}^+_\nu \to \alpha^*_\nu$, $\hat{a}_\mu \to \alpha_\mu$ (for coherent states) of classical amplitudes for the operators $\hat{a}$ and $\hat{a}^+$.

- General many mode coherent states is a direct product of the single mode coherent states:

$$|\{\alpha_n\}\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes |\alpha_3\rangle \dots \otimes |\alpha_n\rangle \otimes \dots \qquad (102)$$

with $$\hat{a}_n|\alpha_n\rangle = \alpha_n|\alpha_n\rangle \qquad (103)$$

Mathematical appendix.

Operator ~~functions~~: identities (Mandell & Wolf Quantum optics, p. 515).

Let $\hat{A}$ and $\hat{B}$ operators.

Commutator relations:
$$\exp[\hat{A}]\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \\ + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \dots \qquad (104)$$

Special case $[\hat{A},\hat{B}] = c$ (constant)

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + c \qquad (105)$$

Similarity transformation $$e^{\hat{A}}f(\hat{B})e^{-\hat{A}} = f(e^{\hat{A}}\hat{B}e^{-\hat{A}}) \qquad (106)$$

Campbell - Baker - Husdorff - Theorem

if $[A,[A,B]] = 0 = [\hat{B},[\hat{A},\hat{B}]$ (107a)

$$\Rightarrow \exp(\hat{A} + \hat{B}) = \exp(\hat{A}) \exp(\hat{B}) \exp[\tfrac{1}{2}[\hat{A},\hat{B}]) =$$

$$= \exp(\hat{B}) \exp(\hat{A}) \exp(\tfrac{1}{2}[\hat{B},\hat{A}]) \quad (107b)$$

Application: Constructing coherent states $|\alpha\rangle$
(for a single mode)

Define: $\hat{T}(\alpha) = \exp(\alpha^* \hat{a} - \alpha \hat{a}^{+})$ (108)

$\Rightarrow$ $\hat{T}^{+}(\alpha) = \exp(\alpha \hat{a}^{+} - \alpha^* \hat{a}) = \hat{T}^{-1}(\alpha) = \hat{T}(-\alpha)$ (109)

$\hat{T}(\alpha)$ is a unitary operator & $\hat{T}(\alpha)\hat{T}^{+}(\alpha) = 1$

using Eq. (104) with $\hat{A} = \alpha^* \hat{a} - \alpha \hat{a}^{+}$; $\hat{B} = \hat{a}$

$\Rightarrow$ $[\hat{A},\hat{B}] = -\alpha[\hat{a}^{+},\hat{a}] = \alpha$

$\Rightarrow$ $\hat{T}(\alpha)\hat{a}\hat{T}^{-1}(\alpha) = e^{\hat{A}}\hat{B}e^{-\hat{A}} \underset{(105)}{=} \hat{a} + \alpha$ (110)

$\Rightarrow$ $\hat{a}\hat{T}^{-1}(\beta)|\alpha\rangle = \hat{T}^{-1}(\beta)\hat{T}(\beta)\hat{a}\hat{T}^{-1}(\beta)|\alpha\rangle =$

$\underset{(110)}{=} \hat{T}^{-1}(\beta)(\hat{a} + \beta)|\alpha\rangle \underset{(103)}{=} \hat{T}^{-1}(\beta)(\alpha + \beta)|\alpha\rangle =$

(111)

$= (\alpha + \beta)\hat{T}^{-1}(\beta)|\alpha\rangle$

special case $\beta = -\alpha$

$a\hat{T}^{-1}(-\alpha)|\alpha\rangle = 0$ (112)

$\Rightarrow \hat{T}^{-1}(-\alpha)|\alpha\rangle \underset{(109)}{=} \hat{T}(\alpha)|\alpha\rangle = \underset{\text{constant}}{k}|0\rangle$ (113)

$$\Rightarrow \quad |\alpha\rangle \underset{(109)(113)}{=} K\,\hat{T}^{+}(\alpha)\,|0\rangle \qquad (114)$$

Normalising : $\langle\alpha|\alpha\rangle = 1 = |K|^2 \qquad K = e^{i\varphi} \qquad (115)$

- Operator create coherent states from vacuum.
- Using Eq. (107) with $\hat{A} = \alpha\hat{a}^{+}$; $\hat{B} = -\alpha^{*}\hat{a}$

$$\Rightarrow \quad [\hat{A},\hat{B}] = |\alpha|^2$$

$$\hat{T}^{+}(\alpha) = e^{\hat{A}+\hat{B}} \underset{(107)}{=} e^{\alpha\hat{a}^{+}} e^{-\alpha^{*}\hat{a}} e^{-\frac{1}{2}\alpha^{*}\alpha} \qquad (116)$$

Since $\hat{a}|0\rangle = 0 \Rightarrow e^{-\alpha^{*}\hat{a}}|0\rangle = |0\rangle \qquad (117)$

$$\Rightarrow \quad |\alpha\rangle \underset{(114)(115)}{=} e^{i\varphi}\,\hat{T}^{+}(\alpha)|0\rangle \underset{(116-117)}{=} e^{i\varphi} e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^{+}}|0\rangle$$

$$= e^{i\varphi} e^{-\frac{1}{2}|\alpha|^2} \sum_{j=0}^{+\infty} \frac{(\alpha\hat{a}^{+})^{j}}{j!}|0\rangle \underset{(71)}{=}$$

$$\underset{(71)}{=} e^{i\varphi} e^{-\frac{1}{2}|\alpha|^2} \sum_{j=0}^{+\infty} \frac{\alpha^{j}}{\sqrt{j!}}|j\rangle \qquad (118)$$

$\Rightarrow$ Probability that a coherent state $|\alpha\rangle$ contains j photons is

$$|\langle j|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2j}}{j!} \qquad (119)$$

$\Rightarrow$ poisson distribution for photon occupation.

$\Rightarrow$ average number of photons

$$\langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^{+}\hat{a}|\alpha\rangle = |\alpha|^2 \qquad (120)$$

square deviation (fluctuations)

$$\langle\alpha|\Delta\hat{N}^2|\alpha\rangle = \langle\hat{N}\rangle = |\alpha|^2 \qquad (121)$$

relative fluctuations

$$\frac{\sqrt{\Delta N^2}}{\langle N\rangle} = \frac{1}{|\alpha|} \qquad (122)$$

$\Rightarrow$ The number of Photons is not defined, relative fluctuations of the number of photons can be larger than 1, quantum effects are expected to be of high intensity.

One obtains (identity relation, completenes)

$$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \sum_{j=0}^{+\infty} |j\rangle\langle j| = \hat{1} \qquad (123)$$

Eq. (123) is obtaine from Eq. (118) where we use explicite integration with $d^2\alpha = d\alpha d\alpha^* = d\mathrm{Re}\,\alpha\, d\mathrm{Im}\,\alpha$

$\Rightarrow$ Using Eq.(123) one can expand any state in terms of the coherent states as

$$|\psi\rangle = \hat{\mathbb{1}}\, |\psi\rangle = \frac{1}{\pi}\int d\alpha^* d\alpha\, |\alpha\rangle\langle\alpha|\psi\rangle \qquad (124)$$

Orthogonality:

$$\langle\beta|\alpha\rangle = e^{i(\varphi_\alpha-\varphi_\beta)}\, e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\, \langle 0| e^{\beta^*\hat{a}}\, e^{\alpha\hat{a}^+}\, \underbrace{e^{-\beta^*\hat{a}}\, \overbrace{e^{\beta^*\hat{a}}\, |0\rangle}^{Eq(117)\ |0\rangle}}_{\hat{1}} =$$

$$\overset{=}{\scriptstyle (106)}\; e^{i(\varphi_\alpha-\varphi_\beta)}\, e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \langle 0| \exp[\underbrace{e^{\beta^*\hat{a}}\,\alpha\hat{a}^+ e^{-\beta^*\hat{a}}}_{\alpha a^+ + \alpha\beta^*\ (105)}]\, |0\rangle$$

$$-\tfrac{1}{2}(-\alpha\beta^* - \alpha^*\beta + \alpha^*\beta - \alpha\beta^*)$$

$$= e^{i(\varphi_\alpha-\varphi_\beta)}\, e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \exp[\cancel{+\alpha\beta^* \underbrace{-\alpha^*\beta + \alpha^*\beta}_{=0} \underbrace{-\alpha\beta^*}_{2i\,\mathrm{Im}(\alpha^*\beta)}}\,]\ \underbrace{\langle 0| e^{\alpha\hat{a}^+}|0\rangle}_{=1}$$

$$\boxed{-\frac{1}{2}(|\alpha|^2+|\beta|^2 - \alpha\beta^* - \alpha^*\beta) = -\frac{1}{2}|\alpha-\beta|^2}$$

$$= \exp[i(\varphi_\alpha - \varphi_\beta - \mathrm{Im}(\alpha^*\beta))]\, e^{-\frac{1}{2}|\alpha-\beta|^2} \neq 0. \qquad (125)$$

- Coherent states are not orthogonal

$\Rightarrow$ Expansion in coheren States is not unique
[Coherent states are overcomplete].

Because a system of states $|\beta\rangle$ can be expanded as

$$|\beta\rangle \underset{(124)}{=} \frac{1}{\pi}\int d\alpha^* d\alpha\ |\alpha\rangle\langle\alpha|\beta\rangle =$$

$$\underset{(125)}{=} \frac{1}{\pi}\int d\alpha^* d\alpha\ |\alpha\rangle \exp\left[-i(\varphi_\alpha-\varphi_\beta-\mathrm{Im}(\alpha^*\beta))-\frac{1}{2}|\alpha-\beta|^2\right]$$

(126)

III. 4. States with minimal uncertainty (Unschärpe) (Squeezed states)

uncertainty relations specify quantum mechanical "errors" in the measured quantities. The best ~~classical limit~~ approximation to the classical theory (without uncertainities) can be achieved by minimizing the uncertainty (fluctuations) of the Hermitian ~~operators~~ $\hat{A}$ and $\hat{B}$.

$$\hat{\alpha} = \hat{A} - \langle\hat{A}\rangle \qquad \beta = \hat{B} - \langle\hat{B}\rangle$$

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle;\ \langle\hat{B}\rangle = \langle\psi|\hat{B}|\psi\rangle$$

Let : $|\psi_\alpha\rangle = \hat{\alpha}|\psi\rangle \qquad |\psi_\beta\rangle = \hat{\beta}|\psi\rangle$

$$\langle\Delta\hat{A}^2\rangle\langle\Delta\hat{B}^2\rangle = \langle\hat{\alpha}^2\rangle\langle\hat{\beta}^2\rangle = \langle\psi_\alpha|\psi_\alpha\rangle\langle\psi_\beta|\psi_\beta\rangle \geq$$

$\hat{\alpha},\hat{\beta}$ are Hermitian

$$\geq |\langle\psi_\alpha|\psi_\beta\rangle|^2 = |\langle\hat{\alpha}\hat{\beta}\rangle|^2 = \left|\frac{1}{2}\langle(\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha})+(\hat{\alpha}\hat{\beta}-\hat{\beta}\hat{\alpha})\rangle\right|^2$$

Shwarz inequality

$$= \left|\frac{1}{2}\underbrace{\langle\{\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}\}\rangle}_{\text{real}} - \frac{i}{2}\underbrace{\langle i[\hat{\alpha},\hat{\beta}]\rangle}_{\text{real}}\right|^2$$

$$\geq \frac{1}{4}|\langle[\hat{\alpha},\hat{\beta}]\rangle|^2$$

$\hat{\alpha},\hat{\beta}$ - Hermitian operators

(127)

Minimal uncertainty is reached when in Shwarz inequality becomes identity, in the last relation in Eq. (127)

$$\Rightarrow \quad \hat{\alpha}|\psi\rangle = |\psi_\alpha\rangle \underset{\uparrow}{=} \varkappa|\psi_\beta\rangle = \varkappa\hat{\beta}|\psi\rangle \qquad (128)$$

"Shwarz equality"

$$\langle\psi|\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}|\psi\rangle = 0 \qquad (129)$$

(last of Eq. (127))

From Eq.(129) with $\hat\alpha|\psi\rangle = x\hat\beta|\psi\rangle$ and $\langle\psi|\hat\alpha =$
$= x^*\langle\psi|\hat\beta$ follows

$$\langle\psi|\hat\alpha\hat\beta + \hat\beta\hat\alpha|\psi\rangle = x^*\langle\psi|\hat\beta^2|\psi\rangle +$$
$$+ x\langle\psi|\hat\beta^2|\psi\rangle = (x+x^*)\langle\psi|\hat\beta^2|\psi\rangle = 0 \qquad (130)$$

If $\langle\hat\beta^2\rangle = 0$ then by $[\hat\alpha,\hat\beta]\neq 0$ then $\langle\hat\alpha^2\rangle \to \infty$
consequently we set

$$x^* + x = 0 \qquad x = i\gamma \qquad \gamma \in \text{Real} \qquad (131)$$

then $$(A - \langle A\rangle)|\psi\rangle = -i\gamma(B - \langle B\rangle)|\psi\rangle$$

$\Longleftrightarrow$ $$(A + i\gamma\hat B)|\psi\rangle = (\langle A\rangle + i\gamma\langle B\rangle)|\psi\rangle$$

Chosing $\gamma = e^{-2r}$

$\Rightarrow$ $$(e^r\hat A + ie^{-r}\hat B)|\psi\rangle = (e^r\langle\hat A\rangle + ie^{-\cancel{2}r}\langle B\rangle)|\psi\rangle \qquad (132)$$

Consequently: a state that minimizes the uncertainty must be an eigenstate of an operator

$$\hat C_r = e^r\hat A + ie^{-r}\hat B \qquad (133)$$

for some real r.

we are interested in such states that minimizes the uncertainty of the pair of operators (canonically conjugat
$\hat q$ and $\hat p$

$\Rightarrow$ $$\hat C_r = e^r\hat q + ie^{-r}\hat p \qquad (134)$$

Consider the following "squeezing operator"

$$\hat S(r) = \exp\left[\frac{r}{2}(\hat a^2 - \hat a^{+2})\right] \qquad (135)$$

$\Rightarrow$ $$\hat S^+(r) = S^{-1}(r) = S(-r) \qquad (136)$$

$\Rightarrow$ S(r) is aunitary operator

One has

$$[\,{}^{r}\!/_{2}\,(\hat{a}^2 - \hat{a}^{+2}), \hat{q}\,] \underset{(40c)(42)}{=} r\,\hat{q} \qquad (137)$$

$$\Rightarrow \hat{S}(r)\,\hat{q}\,\hat{S}(-r) \underset{(104)}{=} \hat{q} + r\hat{q} + \tfrac{1}{2} r^2 \hat{q} + \ldots = \hat{q}\,(1 + r + \tfrac{r^2}{2!} + \ldots$$

$$= \hat{q}\,e^{r} \qquad (138)$$

~~Thus~~ Similarly one obtains

$$\hat{S}(r)\,\hat{p}\,S(-r) = \hat{p}\,e^{-r} \qquad (139)$$

Since $|\psi_{\alpha r}\rangle$ is a state with a minimal uncertainty

$$\underset{(132)(134)}{\Rightarrow} \alpha_r |\psi_{\alpha r}\rangle = \hat{C}_r |\psi_{\alpha r}\rangle \qquad (140)$$

multiplying (140) with $\hat{S}(-r) \Rightarrow$

$$\alpha_r \hat{S}(-r)|\psi_{\alpha r}\rangle \underset{(140)(135)}{=} \hat{S}(-r)\,\hat{C}_r\,S(r)\,\hat{S}(-r)\,|\psi_{\alpha r}\rangle$$

$$\underset{(134)}{=} \Big(e^{r} \underbrace{\hat{S}(-r)\,\hat{q}\,\hat{S}(r)}_{=\hat{q}\,e^{-r}\,(138)} + i e^{-r} \underbrace{\hat{S}(-r)\,\hat{p}\,\hat{S}(r)}_{=\hat{p}\,e^{r}\,(139)}\Big)\hat{S}(-r)\,|\psi_{\alpha r}\rangle$$

$$= (\hat{q} + i\hat{p})\,\hat{S}(-r)\,|\psi_{\alpha r}\rangle = \sqrt{2\hbar}\,\hat{a}\,\hat{S}(-r)\,|\psi_{\alpha r}\rangle \qquad (141)$$

Thus $\hat{S}(-r)|\psi_{\alpha r}\rangle$ is a coherent state with

$$\alpha = \frac{\alpha_r}{\sqrt{2\hbar}} \qquad (142)$$

we denote $|\psi_{\alpha r}\rangle = |\alpha, r\rangle$

$$\Rightarrow |\alpha, r\rangle \underset{(141)(135)}{=} \hat{S}(r)\,|\alpha\rangle \underset{(114)}{=} e^{i\varphi}\,\hat{S}(r)\,\hat{T}^{+}(\alpha)\,|0\rangle \qquad (143)$$

$$\hat{C}_r\,|\alpha, r\rangle = \sqrt{2\hbar}\,\alpha\,|\alpha, r\rangle \qquad (144)$$

$$\Rightarrow \quad \langle r,\alpha|\hat{q}|\alpha,r\rangle \underset{(143)}{=} \langle\alpha|\hat{S}(-r)\hat{q}\,\hat{S}(r)|\alpha\rangle \underset{(138)}{=}$$

$$= e^{-r}\langle\alpha|\hat{q}|\alpha\rangle \underset{(40)(101)}{=} e^{-r}\sqrt{\frac{\hbar}{2}}(\alpha+\alpha^*) \qquad (145)$$

$$\langle r,\alpha|\hat{q}^2|\alpha,r\rangle \underset{(135)(143)}{=} \langle\alpha|\hat{S}(-r)\hat{q}\,\hat{S}(r)\hat{S}(-r)\hat{q}S(r)|\alpha\rangle$$

$$= e^{-2r}\langle\alpha|\hat{q}^2|\alpha\rangle \underset{(42)(40)(101)}{=} e^{-2r}\frac{\hbar}{2}\left((\alpha+\alpha^*)^2+1\right) \qquad (146)$$

Then

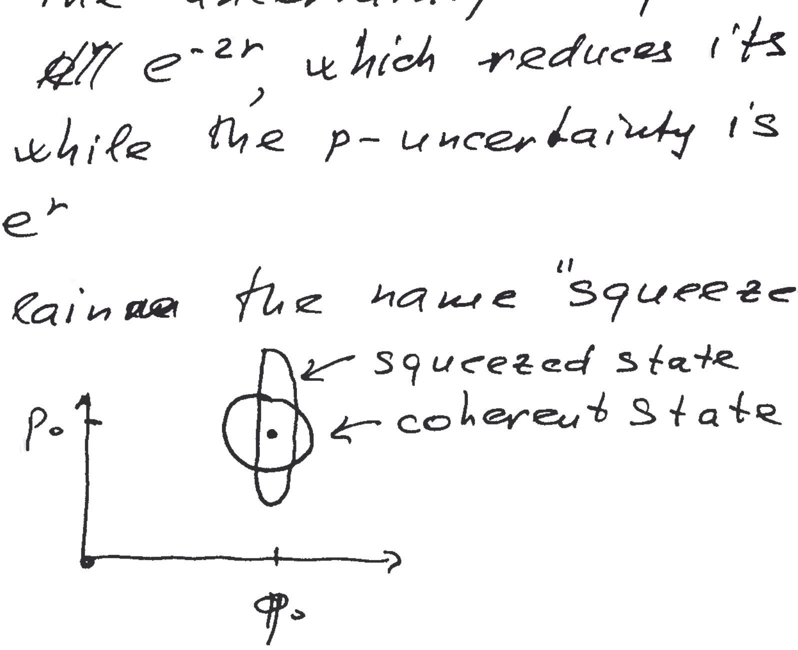
$$\langle r,\alpha|\Delta\hat{q}^2|\alpha,r\rangle = \langle r,\alpha|\hat{q}^2|\alpha,r\rangle - \langle r,\alpha|\hat{q}|r,\alpha\rangle^2 =$$

$$\underset{(145)(146)}{=} e^{-2r}\frac{\hbar}{2} \qquad (147)$$

Similarly: $\langle r,\alpha|\Delta p^2|\alpha,r\rangle = e^{2r}\frac{\hbar}{2}$ (148)

- The minimal uncertainty, as realised by $|\alpha,r\rangle$ states is, as follows also from Eqs. (34), (127)
$$\langle\Delta q^2\rangle\langle\Delta p^2\rangle = \frac{\hbar^2}{4}$$
- If $r=0$ the uncertainty is equal for both q and p.
- For $r\neq0$ the uncertainty in q ~~is multi~~ has a factor e^{-2r}, which reduces it's vacuum value $\hbar/2$, while the p-uncertainty is enhanced by factor e^r

$\Rightarrow$ This explains the name "squeezed states"



III. 5. Thermal (thermalized) light

Contact between the system of fields with ~~energy~~ thermal reservoir at temperature T (for example a contact with a wall with heated gas on the other side)

$\Rightarrow$ A state of the field is described by the statistical operator

$$\hat{\rho} = \prod_n \hat{\rho}_n \qquad (149a)$$

where statistical operator of the mode is given as

$$\hat{\rho}_n = \exp\left[-\frac{\hat{H}_{F_n}}{k_B T}\right]\frac{1}{Z_n} \qquad (149b)$$

k_B - Boltzmann constant, $\hat{H}_{F_n}$ - Eq. (66)

$$H_{F_n} = \hbar\omega_n(\hat{a}_n^+\hat{a}_n + 1/2) \qquad Z_n = S_p\left[e^{-\frac{\hat{H}_F}{k_BT}}\right]$$

$$\Rightarrow \quad \hat{\rho}_n |j_n\rangle = \frac{1}{Z_n}\exp\left[-\frac{\hbar\omega_n}{k_BT}(j_n + 1/2)\right]|j_n\rangle \qquad (150)$$

$\Rightarrow$ $\hat{\rho}_n$ is diagonal in the basis of Fock states

$$\hat{\rho}_n = \sum_{j_n=0}^{+\infty} P_{j_n} |j_n\rangle\langle j_n| \qquad (151)$$

$\hat{\rho}_n$ describes an Ensemble of many with definite number of photons

- The probability of the state $|j_n\rangle$ is given by

$$P_{j_n} = \exp\left[-\frac{\hbar\omega_n}{k_BT}(j_n + 1/2)\right]\frac{1}{Z_n} \qquad (152)$$

Normalization

$$1 = \sum_{j_n} P_{j_n} \Longleftrightarrow \quad Z_n = \sum_{j_n=0}^{+\infty} \exp\left[-\frac{\hbar\omega_n}{k_BT}(j_n + 1/2)\right] =$$

$$= \cancel{\frac{1}{e^{\frac{\hbar\omega_n}{k_BT}} - 1}} \quad \frac{\exp\left[-\frac{\hbar\omega_n}{2k_BT}\right]}{1 - \exp\left[-\frac{\hbar\omega}{k_BT}\right]} \qquad (153)$$

- The number of photons in the statistical ensemble is not defined by quantum mechanics byt by statistical states
- In thermal ensemble quantum eigenstates are energy eigenstates

With Eq. (152) one finds that

$$\langle \hat{N}_n \rangle_T = Sp[\hat{\rho}_n \hat{N}_n] \underset{(151)}{=} \sum_{j_n} j_n P_{j_n} = \frac{1}{e^{\frac{\hbar\omega_n}{k_B T}} - 1} \quad \text{Planck's law} \quad (154)$$

Fluctuations (standard deviation) of the number of photons

$$\langle (\hat{N}_n - \langle \hat{N}_n \rangle)^2 \rangle_T = \sum_j j_n^2 P_{j_n} - \langle N_N \rangle_T^2 =$$

$$= \langle N_N \rangle_T (1 + \langle N_n \rangle_T) \quad (155)$$

⇒ relative standard deviation

$$\frac{\sqrt{\langle \Delta \hat{N}_n^2 \rangle_T}}{\langle N_n \rangle_T} = \sqrt{1 + \frac{1}{\langle N_n \rangle_T}} \quad (156)$$

- relative deviation of the number of photons in thermal equilibrium is > 1
- If the number of photons → ∞ the relative deviation → 1 and not to "0", contrary to the coherent state of light.

IV. Preparation of states of light

IV.1. ~~Expression~~ ~~&~~ ~~Expansions,~~ Production of Fock states

Conceptually one probability to prepare a State $|j_n\rangle$ is via a spontaneous emission of a photon by excited atom. One puts an atom in a resonator, which has its own modes close to the atomic transition modes. The coupling between the field and the atom makes atom to release energy to a ~~mode~~ el.-m. field mode, so a photon is created.

The method requirs careful preparation of atomaic states. Altrernative: Production of Fock stats using parametrical resonance + Idler-measuments.

Process: pump photons (with frequency ω_p) decay into sygnal (ω_s) and Idler (ω_i) photons in an optical non-linear crystall

$$\omega_p = \omega_i + \omega_s \qquad (157)$$

non linear crystall

A simplified model

$$\hat H = \hat H_0 + \hat H_1 \qquad (158a)$$

$$\hat H_0 = \hbar\omega_s \hat a_s^+ \hat a_s + \hbar\omega_i \hat a_i^+ \hat a_i \qquad (158b)$$

$$\hat H_1 = \hbar\left(g e^{-i\omega_p t} \hat a_s^+ \hat a_i^+ + g^* \hat a_s \hat a_i e^{i\omega_p t}\right) \qquad (158c)$$

where $\hat a_s$ and $\hat a_i$ stand for sygnal and idler photons

- The pump field must be in a coherent state of high intencity, so that classical description is a good approximation and the reverse action of the sygnal and Idler photons on E_{pump} is very small and can be neglected.

- The coupling constant $g \sim \chi \cdot E_{pump}$, χ is the succeptibility of the nonlinear crystall.

$\Rightarrow$ Scrödinger representation for time evolution

$$|\psi(t)\rangle = \cancel{e}\,\hat{u}(t)|\psi(t=0)\rangle \qquad (159)$$

$\hat{U}(t)$ - evolution operator, that satisfies

$$i\hbar \frac{\partial}{\partial t}\hat{u} = \hat{H}\hat{U} \qquad (160)$$

$$\hat{u}(t=0) = \hat{1\!\!1} \qquad (161)$$ "Wechselwirkung Bild"

Transformation to "interaction representation"

$$\hat{u} = e^{-i/\hbar H_0 t}\hat{u}_1 \qquad (162)$$

$$\underset{(158)(160)}{\Rightarrow} \quad i\hbar\frac{\partial}{\partial t}\hat{u}_1 = \hat{\tilde{H}}_1 \hat{u}_1 \qquad (163)$$

with

$$\hat{\tilde{H}}_1 = e^{i/\hbar H_0 t} H_1 e^{-i/\hbar H_0 t} =$$

$$\underset{(104)(158c)}{=} \hbar g e^{-i\omega_p t + i(\omega_s+\omega_i)t}\hat{a}_s^+\hat{a}_i^+ + c.c. \underset{(157)}{=} \hbar g\, \hat{a}_s^+\hat{a}_i^+ + c.$$

$\Rightarrow$ $\hat{\tilde{H}}_1$ - does not depend on time

$$\Rightarrow \quad \hat{U}_1(t) = e^{-i/\hbar \tilde{H}_1 t} = \exp[-it(g\hat{a}_s^+\hat{a}_i^+ + g^*\hat{a}_s\hat{a}_i)] =$$

$$= \exp[-\hat{a}_s^+\hat{a}_i^+ e^{i\varphi}\tanh(r)]\exp[-(\hat{a}_s^+\hat{a}_s + \hat{a}_i^+\hat{a}_i)\ln(\cosh(r))]\times$$

$$\times\exp[\hat{a}_s\hat{a}_i e^{-i\varphi}\tanh(r)] \quad \cancel{rtg} \quad re^{i\varphi} = itg \qquad (164a)$$

Details of the derivation are in Barnett, Redmore "Methods in theoretical Quantum optics" page 76.

If no sygnal ~~or~~ Ideler photons exist at $t=0$

then $|\psi(t=0)\rangle = |0\rangle$

$$\underset{(154)(161)(164)}{\Rightarrow} \quad |\psi(t)\rangle = \frac{e^{-i/\hbar H_0 t}}{\cosh(r)} \exp\left[-\hat{a}_s^+ a_i^+ e^{i\varphi} \tanh(r)\right]|0\rangle$$

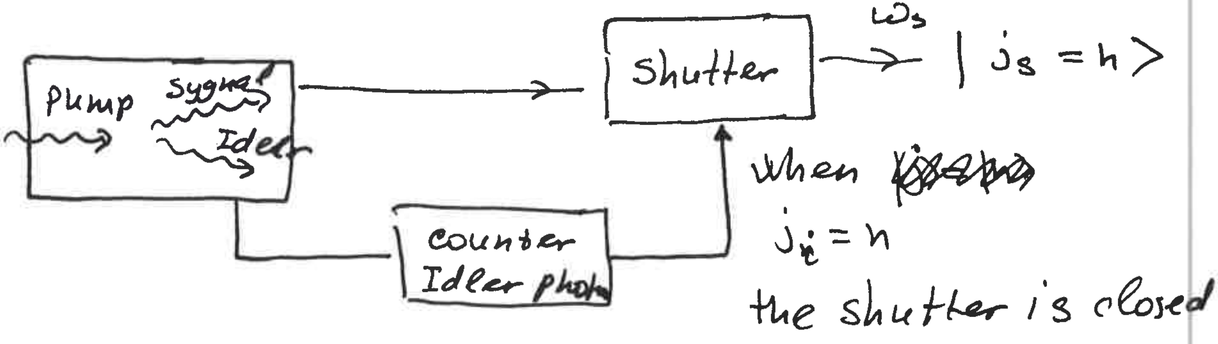
$$= \frac{e^{i/\hbar H_0 t}}{\cosh(r)} \sum_{j=0}^{+\infty} \left[-e^{i\varphi} \tanh(r)\right]^j \frac{\left(\hat{a}_s^+ \hat{a}_i^+\right)^j}{j!} |0\rangle =$$

$$\underset{(74)(75)}{=} \frac{1}{\cosh(r)} \sum_{j=0}^{+\infty} \left(-e^{i(\varphi-(\omega_s+\omega_i))t} \tanh(r)\right)^j |j_s=j\rangle \otimes |j_i= \qquad (165)$$

The number of Ideler photons and the number of sygnal photons is the same in each component

$\Rightarrow$ Measuring the number of Idler photons set nu number of sygnal photons ~~tho~~ the same value.

A schematic diagraeem to prepare photons



IV 2. Production of coherent states.

Statement: coupling with a given classical current produces a coherent state, if the field is initially in a vacuum ~~field~~ state

Consider a single mode (we omit mode indices) as in Eq. (41), described by operator $\hat{a}$ that satisfy the Heisenberg equation

$$\dot{\hat{a}} = -\omega i a - \frac{\dot{J}}{\sqrt{2\hbar}} \qquad (166).$$

Its solution is $$\hat{a}(t) = e^{-i\omega t}\hat{a}(t=0) + \alpha(t) \qquad (167a)$$

where $$\alpha(t) = -\frac{1}{\sqrt{2\hbar}}\int_{t_0}^{t} j(t')\, e^{-i\omega(t-t')}\, dt' \qquad (167b)$$

In the Heisenberg representation the state [illegible] is time independent

$$|\Psi_H\rangle = |0\rangle \qquad (168)$$

H – for Heisenberg

Then $$\hat{a}(t)|\Psi_H\rangle \underset{(167)(168)}{=} e^{-i\omega t}\underbrace{\hat{a}(t=t_0)|0\rangle}_{=0} + \alpha(t)|0\rangle = \alpha(t)|\Psi_H\rangle \qquad (169)$$

In order to change to the Schrödinger representation we use the evolution operator

$$|\psi_s(t)\rangle = \hat{U}(t,t_0)|\psi_s(t=0)\rangle = \hat{U}(t,t_0)|\psi_H\rangle \quad (170a)$$

Schödinger representation for operators

$$\hat{a}_s = \hat{U}(t,t_0)\,\hat{a}_H(t)\,\hat{U}^{+}(t,t_0) = \hat{U}\,\hat{a}\,U^{+} \quad (170b)$$

Further

$$\alpha(t)|\psi_s(t)\rangle \underset{(170)}{=} \alpha(t)\{\hat{U}|\psi_H\rangle \underset{(169)}{=} \hat{U}\,\hat{a}_H|\psi_H\rangle =$$

$$\underset{UU^{+}=1}{=} \underbrace{\hat{U}\,\hat{a}_H\hat{U}^{+}}_{\hat{a}_s\,(170)}U|\psi_H\rangle \underset{(170)}{=} \hat{a}\,\hat{U}|\psi_H\rangle = \hat{a}|\psi_s\rangle \quad (171)$$

So, $|\psi_s(t)\rangle$ is an eigenstate of $\hat{a}$ with an eigenvalue $\alpha(t)$ $\Rightarrow$ $|\psi_s(t)\rangle$ is a coherent state.

Remarks: • The model for the field-current coupling is valid if

a) The back action of the field to the current is negligible

b) Quantum fluctuations of the current are negligible → the current is purely classical.

- In the formulation of classical electrodynamics current generates field, as follows from ~~the~~ equation (29) for the modes. If the current is given externally then ~~the~~ Eq. (29) gives (with the condition

$$\alpha(t=t_0)=0$$ – no field befor the current is switched on)

$$\alpha(t) = -\int_{t_0}^{t} dt' \, j_n(t') \, e^{-i\omega_n (t-t')} \qquad (172)$$

Taking Eqs. (172), (67b) with (94b), (59) shows that in Eq. (171) coherent states are constructed as a quasiclassical state, as described in III.3 where the classical field is ~~described~~ constructed from the current.

- In practice, laser light is a good example of a photon state (ensemble of photons) which is close to a coherent state.

IV. 3. Construction of squeezed states.

Statement: squeezed states are constructed via "degenerate parametrical resonance": in which one Pump photon (2ω) decays into two identical photons (frequency ω) (decay is arranged by a non-linear optical crystall)

The modell is analogous to the model in Eq (158)

$$\hat H = \hbar\omega\,\hat a^+\hat a + i\hbar\lambda\left(\hat a^{+2}e^{-2i\omega t} - \hat a^2 e^{2i\omega t}\right) \quad (173)$$

The pump field is classical, the coefficient λ is assumed real

=> Heisenberg time evolution

$$\dot{\hat a} = -i\omega\hat a + 2\lambda\hat a^+ e^{-2i\omega t} \qquad (174)$$

substitute

$$\hat b = e^{i\omega t}\hat a \quad (175) \Rightarrow \dot{\hat b} = 2\lambda\hat b^+ \quad (176a)$$

$$\dot{\hat b}^+ = 2\lambda\hat b \quad (176b)$$

$\hat b$ and $\hat b^+$ have the same algebra (commutators) as $\hat a$ and $\hat a^+$. Also, $\hat b|0\rangle = 0$ and $|j\rangle = e^{ij\omega t}\frac{\hat b^{+j}}{\sqrt{j!}}|$

so $\hat b$ and $\hat b^+$ acts analogously to $\hat a$ and $\hat a^+$ to create and destroys the modes. For the associated observables one writes

$$\hat b_q = \sqrt{\frac{\hbar}{2}}\left(\hat b + \hat b^+\right) \qquad (177a)$$

$$\hat b_p = \frac{1}{i}\sqrt{\frac{\hbar}{2}}\left(\hat b - \hat b^+\right) \qquad (177b)$$

From (176)

$$\begin{cases} \dot{\hat{b}}_q = \sqrt{\frac{\hbar}{2}}\,(\dot{\hat{b}}_q + \dot{\hat{b}}_q^+) \underset{(176)}{=} 2\lambda \xi_2 \sqrt{\frac{\hbar}{2}}\,(\hat{b} + \dot{\hat{b}}^+) = 2\lambda \hat{b}_q & (178a) \\ \dot{\hat{b}}_p = -2\lambda \dot{\hat{b}}_q & (178b) \end{cases}$$

$$\Rightarrow \begin{cases} \hat{b}_q(t) = e^{2\lambda t}\,\hat{b}_q(t=0) & (179a) \\ \hat{b}_p(t) = e^{-2\lambda t}\,\hat{b}_p(t=0) & (179b) \end{cases}$$

Denote the squeezing parameter $r = -2\lambda t$ ~~(182)~~

and write an analog to Eq. (133) and (134)

$$\hat{C}_r(t) = e^r \hat{b}_q(t) + i e^{-r} \hat{b}_p(t) =$$

$$\underset{(180)(179)}{=} \hat{b}_q(t=0) + i \hat{b}_p(t=0) =$$

$$\underset{(177)}{=} \sqrt{2\hbar}\,\hat{b}(t=0) = \sqrt{2\hbar}\,\hat{a}_s \qquad (180)$$

$\Rightarrow$ the system is initially in a coherent state $|\alpha\rangle$ (in practice, usually in vacuum)

$\Rightarrow |\Psi_H\rangle = |\alpha\rangle$

$$\Rightarrow \hat{C}_r(t)|\Psi_H\rangle \underset{(99)(181)}{=} \sqrt{2\hbar}\,\alpha|\Psi_H\rangle \underset{\text{as in }(142)}{=} \alpha_r|\Psi_H\rangle \qquad (181)$$

multiplying with $\hat{U}(t) = \hat{U}(t, t_0)$

$$\Rightarrow \hat{C}_{r_S}|\Psi_S\rangle = \alpha_r|\Psi_S\rangle \qquad (182)$$

where $|\Psi_S\rangle = \hat{U}(t)|\Psi_H\rangle$ and $\hat{C}_{rS} = \hat{U}(t)\hat{C}_r\hat{U}^+(t) =$

$$= e^r \hat{b}_{qS} + i e^{-r} b_{pS} \qquad (183)$$

Then $|\Psi_s\rangle$ is the state which minimises the uncertainty of the operator pair $\hat{b}_{p_s}$ and $\hat{b}_{q_s}$.

Explicit form of the $\hat{b}_{ps}$ and $\hat{b}_{qs}$ in the Schrödinger representation

$$\hat{b}_{q_s} = \hat{U}(t)\,\hat{b}_q(t)\,\hat{U}^+(t) \underset{(177a)}{=} \sqrt{\frac{\hbar}{2}}\,\hat{U}(t)\left(\hat{b}(t)+\hat{b}^+(t)\right)\hat{U}^+(t)$$

$$\underset{(175)}{=} \sqrt{\frac{\hbar}{2}}\left(e^{i\omega t}\hat{a}_s + e^{-i\omega t}\hat{a}_s^+\right) \qquad (184a)$$

$$\hat{b}_{ps} = \frac{1}{i}\sqrt{\frac{\hbar}{2}}\left(e^{i\omega t}\hat{a}_s - e^{-i\omega t}\hat{a}_s^+\right) \qquad (184b)$$

Construction of $|\Psi_s\rangle$ is analogous to the construction procedure of the squeezed states $|\alpha, r\rangle$ according to (143) and (144), Eq. (143) defines the squeezed state by action of ~~the~~ operator $\hat{S}(r)$ (defined in Eq. (134 and ~~the~~ operator $\hat{T}^+(\alpha)$ (in Eq. (103)). ~~[illegible]~~ Here Were we use ~~the~~ Substitution

$$\hat{a} \rightarrow \hat{b} = e^{i\omega t}\hat{a}_s$$
$$\hat{a}^+ \rightarrow \hat{b}^+ = e^{-i\omega t}\hat{a}_s^+$$

V. Application: beam spliting.

Beam splitter: half transparent mirror at $z=0$ (BS).

The goal is to describe BS by the modified mode System. (Eq. 19a)

Strategy: Consider modes with different coefficients at $z>0$ and $z<0$.

=> Eq. (18) for the modes can be solved for respective Half space if we take waves with the same wave vector amplitude $|k|$.

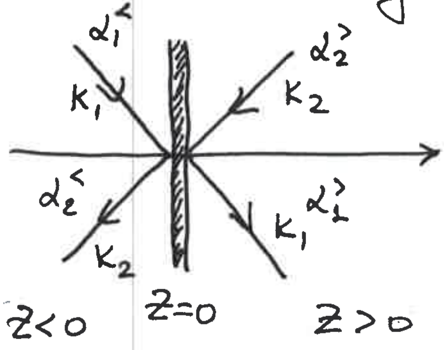
=> Connection be-tween coefficients follow from $\boxed{\text{Maxwell equations}}$ and the boundary conditions for they depend from the BS material.

We set the following conditions:

a) BS is a linear media (no higher hammonics)

b) Same polarization of all modes (connected), so that λ index will be considered same and omitted.

=> Participating modes: 1) one incoming $\vec{k}_1$ mode, and transmitted $\vec{k}_1$ mode and reflected $\vec{k}_2$ mode 2) one incoming $\vec{k}_2$ mode, and transmitted $\vec{k}_2$ mode and reflected $\vec{k}_1$ mode



Total mode:

$$\vec{e}(r) = \begin{cases} \alpha_1^< \vec{e}_1(r) + \alpha_2^< \vec{e}_2(r), & z<0 \\ \alpha_1^> \vec{e}_1(r) + \alpha_2^> \vec{e}_2(r), & z>0 \end{cases} \qquad (185)$$

=> Boundary conditions couples these four coefficients

$$\begin{bmatrix}\alpha_1^{>}\\ \alpha_2^{<}\end{bmatrix} = U\begin{pmatrix}\alpha_1^{<}\\ \alpha_2^{>}\end{pmatrix} = \begin{pmatrix}T_1 & R_2\\ R_1 & T_2\end{pmatrix}\begin{pmatrix}\alpha_1^{<}\\ \alpha_2^{>}\end{pmatrix} \qquad (186)$$

T_1, T_2 – transmission coefficients, R_1, R_2 – reflection coefficient

c) BS is conservative system: does not loos. energy

<=> (91) $|\alpha_1^{>}|^2 + |\alpha_2^{<}|^2 = |\alpha_1^{<}|^2 + |\alpha_2^{>}|^2$ (187)

=> matrix $\hat{U}$ is unitary, $U^+U = \hat{1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (188)

V. 1. Input (incoming) modes (satisfy a) b) c))

=> Normalization: $|\alpha_1^{<}|^2 + |\alpha_2^{>}|^2 = 1$ (189)

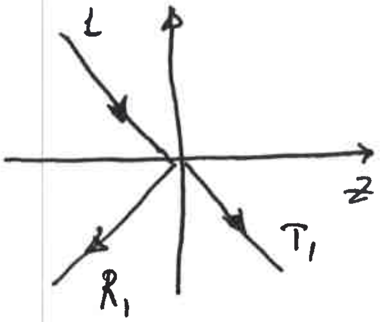
=> output modes are found from Eq. (186)

=> Basis states can be chosen $\vec{e}_{I_1}$ and $\vec{e}_{I_2}$ so that

I_1: $\alpha^{<}_{1I_1} = 1$; $\alpha^{>}_{2I_1} = 0$

$\alpha^{>}_{1I_1} = T_1$; $\alpha^{<}_{2I_1} = R_1$

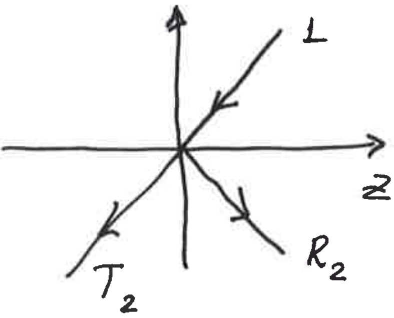
$\vec{e}_{I_1}$



I_2: $\alpha^{<}_{1I_2} = 0$; $\alpha^{>}_{2I_2} = 1$

$\alpha^{>}_{1I_2} = R_2$; $\alpha^{<}_{2I_2} = T_2$

$\vec{e}_{I_2}$



$\vec{e}_{I_1}$ and $\vec{e}_{I_2}$ are degenerate $\omega_{I_1} = \omega_{I_2} = \omega$

$$\begin{pmatrix}T_1\\ R_1\end{pmatrix} = \begin{pmatrix}T_1 & R_2\\ R_1 & T_2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$\begin{pmatrix}T_2\\ R_2\end{pmatrix} = \begin{pmatrix}T_1 & R_2\\ R_1 & T_2\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

V. 2. Output modes (outgoing)

Reversing (186) gives

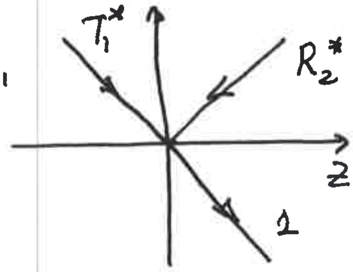
$$\begin{pmatrix} \alpha_1^{<} \\ \alpha_2^{>} \end{pmatrix} = U^{+} \begin{pmatrix} \alpha_1^{>} \\ \alpha_2^{<} \end{pmatrix} = \begin{pmatrix} T_1^* & R_1^* \\ R_2^* & T_2^* \end{pmatrix} \begin{pmatrix} \alpha_1^{>} \\ \alpha_2^{<} \end{pmatrix} \qquad (190)$$

$\Rightarrow$ Basis for output modes $\vec{e}_{O_1}$ and $\vec{e}_{O_2}$

O_1: $\alpha^{>}_{1O_1} = 1$; $\alpha^{<}_{2O_2} = 0$

$\alpha^{<}_{1O_1} = T_1^*$; $\alpha^{>}_{2O_1} = R_2^*$

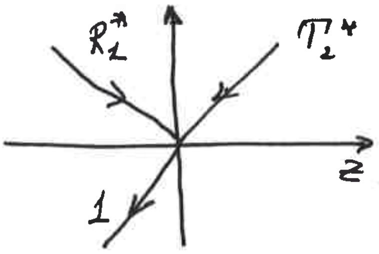
$\vec{e}_{O_1}$



$\alpha^{>}_{1O_2} = 0$; $\alpha^{<}_{2O_2} = 1$

$\alpha^{<}_{1O_2} = R_1^*$; $\alpha^{>}_{2O_1} = T_2^*$

$\vec{e}_{O_2}$



$\Rightarrow$ Change in the basis according to

$$\vec{e}_{O_1} = T_1^* \vec{e}_{I_1} + R_2^* \vec{e}_{I_2} \qquad (191~~3~~a)$$

$$\vec{e}_{O_2} = R_1^* \vec{e}_{I_1} + T_2^* \vec{e}_{I_2} \qquad (192~~3~~b)$$

(You can prove (191) by substituting the definition of the modes and using unitarity as in Eq. (188)).

$\Rightarrow$ $\omega_{O_1} = \omega_{O_2} = \omega$

V.3. Field operators for beam splitted modes.

Field operators can be constracted according to Eq. (51) for both input and output modes,

we write

$$\hat{\vec{E}}^{(-)} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0}} \left(\hat{a}^+_{I_1} \vec{e}^{\,*}_{I_1}(r) + \hat{a}^+_{I_2} \vec{e}^{\,*}_{I_2}(r) \right) =$$

$$= \sqrt{\frac{\hbar\omega}{2\varepsilon_0}} \left(\hat{a}^+_{O_1} \vec{e}^{\,*}_{O_1}(r) + \hat{a}^+_{O_2} \vec{e}^{\,*}_{O_2}(r) \right)$$

$$\underset{(191)}{=} \sqrt{\frac{\hbar\omega}{2\varepsilon_0}} \Big\{ \hat{a}^+_{O_1} \left(T_1 e^{*}_{I_1}(r) + R_2 \vec{e}^{\,*}_{I_2}(r) \right) + \hat{a}^+_{O_2} \left(R_1 e^{*}_{I_1}(r) + T_2 \vec{e}^{\,*}_{I_2}(r) \right) \Big\} \qquad (192)$$

$$\begin{pmatrix} \hat{a}^+_{I_1} \\ \hat{a}^+_{I_2} \end{pmatrix} = \begin{pmatrix} T_1 & R_1 \\ R_2 & T_2 \end{pmatrix} \begin{pmatrix} \hat{a}^+_{O_1} \\ \hat{a}^+_{O_2} \end{pmatrix} \qquad (193)$$

V.4. Quantum states of the field with the beam splitter

Situation: incoming waves are coming as input modes to the beam splitter from directions $\vec{k}_1, \vec{k}_2$ and the sygnal is detected {by detecting output modes} O_1, O_2.

1. Example: the beam with a photon state that contains $|n\rangle$ photons comes from direction $\vec{k}_1$.

$$\Rightarrow |\Psi_{nk_1}\rangle \underset{(74)}{=} \frac{a^{+n}_{I_1}}{\sqrt{n!}} |0\rangle = |n\rangle_{I_1} \otimes |0\rangle_{I_2} =$$

$$\underset{(193)}{=} \frac{1}{\sqrt{n!}} \left(T_1 \hat{a}^+_{O_1} + R_1 \hat{a}^+_{O_2} \right)^n |0\rangle = \frac{1}{\sqrt{n!}} \sum_{j=0}^{n} \binom{n}{j} T_1^{\,j} R_1^{\,n-j} \hat{a}^{+j}_{O_1} \hat{a}^{+(n-j)}_{O_2} |0\rangle$$

$$\underset{(74)}{=} \sum_{j=0}^{n} \sqrt{\binom{n}{j}}\, T_1^{\,j} R_1^{\,n-j} |j\rangle_{O_1} \otimes |n-j\rangle_{O_2} \qquad (194)$$

$$\binom{n}{j} = \frac{n!}{(n-j)!\,j!}$$

A special example : 1 photon state

$$|\psi_{1k_1}\rangle = T_1 |1\rangle_{O_1} \otimes |0\rangle_{O_2} + R_1 |0\rangle_{O_1} \otimes |1\rangle_{O_2} \qquad (195)$$

is the prototype of a "Scrödinger cat" state.

=> The photon ~~is~~ simultaneously (at the same time) ~~is the~~ is going to ~~[illegible]~~ ~~with~~ $\vec{k}_1$ ($|1\rangle_{O_1}\otimes|0\rangle_{O_2}$) and ~~the~~ in k_2 ($|0\rangle_{O_1}\otimes|1\rangle_{O_2}$) directions. If one reflects the beams into a single point the interference picture can be detected. However, if one measures the photon in one beam, the measuring ~~of the~~ of the other beam ~~p~~ will produce "0" photons. Measurements will destroy coherence via the reduction of the state.

2. Example 2: incoming wave in a coherent state comes from direction $\vec{k}_1$.

$$\Rightarrow |\psi_{\alpha k_1}\rangle = |\alpha\rangle_{I_1}\otimes|0\rangle_{I_2} = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{+\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle_{I_1}\otimes|0\rangle_{I_2} =$$

$$\underset{(194)}{=} e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{+\infty}\sum_{j=0}^{n}\frac{(T_1\alpha)^j}{\sqrt{j!}}\overset{(118)}{\frac{(R_1\alpha)^{n-j}}{\sqrt{(n-j)!}}}|j\rangle_{O_1}\otimes|n-j\rangle_{O_2} =$$

$$= e^{-\frac{1}{2}|\alpha|^2}\sum_{l,m=0}^{+\infty}\frac{(T_1\alpha)^l}{\sqrt{l!}}|l\rangle_{O_1}\otimes\frac{(R_1\alpha)^m}{\sqrt{m!}}|m\rangle_{O_2} =$$

$$\underset{(118)}{=} e^{-\frac{1}{2}|\alpha|^2}e^{\frac{1}{2}|T_1\alpha|^2}e^{\frac{1}{2}|R_1\alpha|^2}|T_1\alpha\rangle_{O_1}\otimes|R_1\alpha\rangle_{O_2} =$$

$$\underset{(188)}{=} |T_1\alpha\rangle_{O_1}\otimes|R_1\alpha\rangle_{O_2} \qquad (196)$$

=> $|\psi_{\alpha k_1}\rangle$ is not entangled in the output basis, coherent input give a coherent output in both channels. The amplitudes of the corresponding quantum states are the same as ~~is~~ found classically.

=> Initially strong coherent state. (with large intensity) can be made weaker and weaker by ~~a~~ repetitive beam splitting.

3. Example: 1 photon beam from $\vec{K}_1$ direction
+ 1 photon beam from $\vec{K}_2$

$$|\Psi_{K_1K_2}\rangle = |1\rangle_{I_1}\otimes|1\rangle_{I_2} = \hat{a}^+_{I_1}\hat{a}^+_{I_2}|0\rangle =$$

$$\underset{(193)}{=} (T_1\hat{a}^+_{0_1} + R_1\hat{a}^+_{0_2})(R_2\hat{a}^+_{0_1} + T_2\hat{a}^+_{0_2})|0\rangle =$$

$$= \{T_1R_2\hat{a}^{+2}_{0_1} + R_1T_2\hat{a}^{+2}_{0_2} + (R_1R_2+T_1T_2)\hat{a}^+_{0_1}\hat{a}^+_{0_2}\}|0\rangle \quad (197)$$

from unitary condition (Eq. (188)) follows

$$T_1R_2^* = -T_2^*R_1 \quad (198)$$

$$\Rightarrow (R_1R_2+T_1T_2)R_2^* = R_1|R_2|^2 + T_2T_1R_2^* \underset{(198)}{=} R_1(|R_2|^2-|T_2|^2) \quad (199)$$

in a spesial case 50:50 splitting $|R_2|^2 = |T_2|^2 = 1/2$ (200)

(199)(200) $\Rightarrow$ $R_1R_2 + T_1T_2 = 0$ (201)

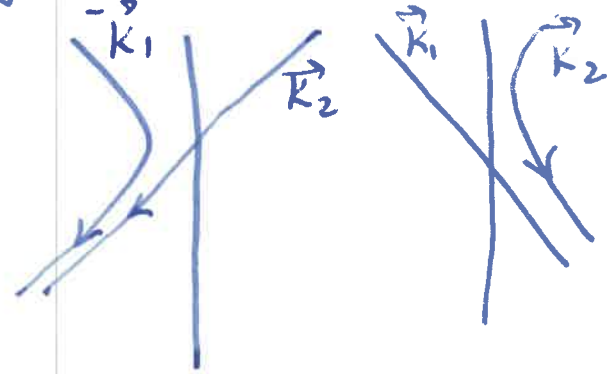
$$\underset{(197)}{\Rightarrow} |\Psi_{K_1K_2}\rangle = \sqrt{2}(T_1R_2|2\rangle_{0_1} + R_1T_2|2_0\rangle_{0_2}) \quad (202)$$

Eq. (188) gives $|T_1| = |T_2|$ $|R_1| = |R_2|$ implicitely

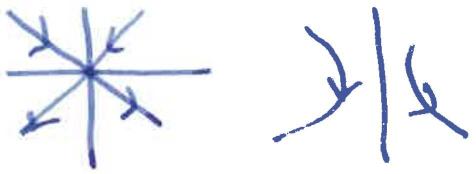
then Eq. (202): a) one has equal probability of detecting 2 photons in $\vec{K}_1$ direction or 2 photons in $\vec{K}_2$ direction

b) there will be no coincidence: if a deqtector counts a photon in one direction, there will be no plotons in other direction

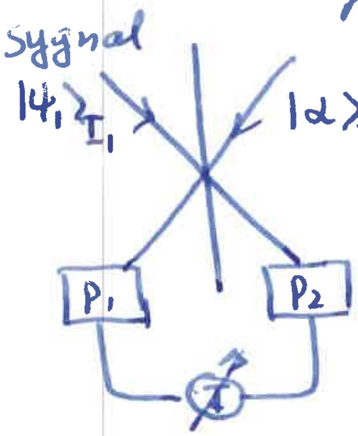
Only two possibilities



The processes are forbidden



4. Example: Homodyn detection with the ~~balancing [illegible]~~ balance



Coherent field

mirror 50:50

P_j -photodetectors for photocurrent

$I_j \sim \langle \hat{a}^+_{0_j} a_{0_j} \rangle$

Balancing sygnal : $I = I_1 - I_2 \sim \langle \hat{a}^+_{0_1} a_{0_1} \rangle - \langle \hat{a}^+_{0_2} \hat{a}_{0_2} \rangle$ (203

Assume 50:50 splitter with

$$T_1 = T_2 = R_2 = -R_1 = \frac{1}{\sqrt{2}} \qquad (204)$$

Substituting this into Eq. (193) gives with (204)

$$\begin{pmatrix} \hat{a}^+_{0_1} \\ \hat{a}^+_{0_2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}^+_{I_1} \\ \hat{a}^+_{I_2} \end{pmatrix} \qquad (205)$$

$$\Rightarrow \hat{a}^+_{0_1} \hat{a}_{0_1} = \frac{1}{2} (\hat{a}^+_{I_1} \hat{a}_{I_1} + \hat{a}^+_{I_2} \hat{a}_{I_2} + \hat{a}^+_{I_1} \hat{a}_{I_2} + \hat{a}_{I_2} \hat{a}_{I_1}) \qquad (206a)$$

$$\hat{a}^+_{0_2} \hat{a}_{0_2} = \frac{1}{2} (\hat{a}^+_{I_1} \hat{a}_{I_1} + \hat{a}^+_{I_2} \hat{a}_{I_2} - \hat{a}^+_{I_1} \hat{a}_{I_2} - \hat{a}^+_{I_2} \hat{a}_{I_1}) \qquad (206b)$$

$$\Rightarrow \hat{I} = (\hat{a}^+_{0_1} \hat{a}_{0_1} - \hat{a}^+_{0_2} \hat{a}_{0_2}) \sim \hat{a}^+_{I_1} \hat{a}_{I_2} + \hat{a}^+_{I_2} \hat{a}_{I_1} \qquad (207)$$

With the field state $|\psi\rangle = |\psi_1\rangle_{I_1} \otimes |\alpha\rangle_{I_2}$ (208)

and $\alpha = |\alpha| e^{i\varphi}$ (209)

$$\underset{(207)}{\Rightarrow} \langle \psi | I | \psi \rangle = |\alpha| \; {}_{I_1}\langle \psi_1 | e^{i\varphi} \hat{a}^+_{I_1} + e^{-i\varphi} \hat{a}_{I_1} | \psi_1 \rangle_{I_1} \qquad (210)$$

$\Rightarrow$ • for each choice of φ one measures the <u>quadrature</u> components of the $|\psi_1\rangle$ field

for example

$$\varphi = 0 \longrightarrow \langle \psi_1 | I | \psi_1 \rangle \sim {}_{I_1}\langle \psi_1 | q_{I_1} | \psi_1 \rangle_{I_1} \qquad (211)$$

$$\varphi = \pi/2 \longrightarrow \langle \psi | I | \psi \rangle \sim {}_{I_1}\langle \psi_1 | \hat{p}_{I_2} | \psi_1 \rangle_{I_2} \qquad (212)$$

p_{I_1}

VI. $\boxed{V}$ Interaction between el-m fiels and matter.

Modell for matter: N – point-like particles at positions ~~x_j~~ r_j ~~und~~ masses m_j, and charges q_j

particle density: $\rho(x,t) = \sum_{j=1}^{N} q_j\, \delta(\vec{x} - \vec{x}_j)$ (172a) (213a) (203

current density: $\vec{j}(x,t) = \sum_{j} q_j\, \dot{\vec{r}}_j\, \delta(x - \vec{r}_j)$ (173b) (213b) (20

Description of the system field + particles

a) Maxwell equation ~~for~~ (1) for fields with ρ and j

b) Lorenz forces for particles $m_j \ddot{\vec{r}}_j = q_j \{\vec{E}(r_j,t) + \dot{\vec{r}}_j \times \vec{B}$

Quantization strategy: 1) Hamiltonian function for ~~[illegible]~~ the total system, represented by pairs of canonically conjugate operators for particles and fields

2) Canonical variables → operators with commutator relations

We already have: decomposition (Eq(4)) $E = E_W + E_Q$

E_Q is defined by material parameters.

Now aim is: to substitute E_Q in the energy of the field

~~to express E_Q in the form of energy and~~

$$E_F = \int d^3x \left(\frac{\varepsilon_0}{2} \vec{E}^2 + \frac{1}{2\mu_0} \vec{B}^2 \right) \underset{(60)}{=} \int d^3x \left(\frac{\varepsilon_0}{2} \bar{E}_Q^2 + \frac{\varepsilon_0}{2} \bar{E}_W^2 + \right.$$

$$\left. + \frac{1}{\mu_0} \bar{B}^2 \right) \overset{(174)}{\Longrightarrow} \text{Longitudinal (sourcess) field}$$

$$H_L = \int d^3x \frac{\varepsilon_0}{2} \bar{E}_Q^2 \underset{(12)}{=} \int d^3x \frac{\varepsilon_0}{2} (\nabla V_c) \cdot (\nabla V_c) = - \frac{\varepsilon_0}{2} \int d^3x\, \Delta V_c \cdot V_c =$$

$$\underset{(11)}{=} \int d^3x\, V_c(x) \rho(x,t) \underset{(12)}{=} \frac{1}{2} \int d^3x\, d^2y \frac{\rho(x,y)\rho(x,y)}{4\pi\varepsilon_0 |x-y|}$$ (175) (214) (205)

with (213a) =>

(173a) (176a)

$$H_L = V_{coul}(\vec{r}_1, \vec{r}_2 \dots \vec{r}_N) + E_{self} .$$ (215) (206)

$$V_{coul} = \frac{1}{2} \sum_{j \neq j'} \frac{q_j q_{j'}}{4\pi\varepsilon_0 |\vec{r}_j - \vec{r}_{j'}|}$$ (216) (76b) (207)

E_{self} is obtained by $j = j'$ and is infinite, but it is constant and is irrelevant

$\vec{E}_{c_j} = -\nabla_{\vec{r}_j} V_{coul}$ = coulomb field that acts on a particle j. $\vec{E}_{c_j}$ describes all longitudinal fields that affects the particles.

using coulomb gauge (Eichung) for the vector potential $\vec{A}(x,t)$

$$\Rightarrow \begin{cases} \vec{E}(r_j,t) = -\dot{\vec{A}} - \vec{\nabla}_{\vec{r}_j} V_{coul} & (217) \quad (177a)(208 \\ \vec{B}(r_j,t) = \vec{\nabla}_{r_j} \times \vec{A} & (218) \quad (177b)(20 \\ \operatorname{div} \vec{A} = 0. & (219) \quad (177c)(21 \end{cases}$$

=> Lorentz forces can be inserted into the Hamiltonian fuction as

(178)

$$H_{Materie} = \sum_j \frac{1}{2m_j} (P_j - q_j \vec{A}(r_j,t)) + V_{coul} .$$ (220) (211)

The Lorenz forces appear in the Hamilton equation

(179)

$$\dot{\vec{r}}_j = \frac{\partial H_M}{\partial \vec{P}_j} \equiv \nabla_{\vec{P}_j} H \; ; \quad \dot{\vec{P}}_j = -\nabla_{\vec{r}_j} H_M$$ (221) (212)

$\vec{P}_j$ is a canonical conjugate momentum to $\vec{r}_j$

Note: velocity of a particle

(180)

$$\dot{\vec{r}}_j = \frac{1}{m_j} (P_j - q_j A(r_j,t)) = \frac{1}{m_j} P_{j,kin}$$ (222) (213)

$P_{j,kin}$ — kinetic momentum.

$$\Rightarrow H_{matter} = H_0 + \underset{I}{H_{ww}}$$ (181a)

$$H_0 = \sum_j \frac{\vec{p}_j^{\,2}}{2m_j} + V_{coul}$$ (214) (223) (181b)

$$H_{ww} = \sum_j \left[-\frac{q_j}{m_j}\vec{p}_j \cdot \vec{A}(\vec{r}_j,t) + \frac{q_j^2}{2m_j}\vec{A}^2(\vec{r}_j,t) \right]$$ (215) (224) (181c)

- In the Coulomb gauge E_Q is a part of electric field in ~~material~~ system, the state of the material system is obtained using H_0 and the interactio Hamiltonian between particle and transverse field
- H_{ww} accounts for the interaction of matter and fields.
- H_{ww} can be expressed using independent field variables, for example using mode decomposition of $\vec{A}(r,t)$.

Since $\operatorname{div} A = 0$, $\vec{A}$ can be expressed using tranvese modes as

$$\vec{A}(\vec{r},t) = \sum_n a_n \vec{e}_n(r)$$ (216) (225) (182)

$$\Rightarrow \vec{B}(r,t) = \operatorname{rot}\vec{A} \underset{(225)}{=} \sum a_n(t) \underbrace{\operatorname{rot}\vec{e}_n}_{(18)\ \frac{\omega_n}{c}\vec{b}_n} \underset{(248)}{=} \sum_n \sqrt{\mu_0\omega_n}\, q_n(t)\vec{b}_n$$ — (183) (217

$$\Rightarrow a_n(t) = \frac{1}{\sqrt{\omega_n \epsilon_0}}\, q_n(t)$$ — (184) (218)

Similarly to Eq. (50) - (52) one obtains — (226) (185a) (219)

$$\vec{A} = \vec{A}^{(+)} + \vec{A}^{(-)}$$ (227) (185b) (220)

$$A^{(+)}(r,t) = \sum_n \frac{1}{2i\sqrt{\epsilon_0\omega_n}} (q_n(t) + ip_n(t))\, e_n(r)$$ — (228) (186) (221)

$$A^{(-)}(r,t) = (A^{(+)}(r,t))^*$$

- In order to write the energy of a compete system field + matter we have to collect all energy (222)

$$H = H_{mat} + H_F = H_0 + \underset{I}{H_{ww}} + H_F$$ (229) (187)

$$H_F \underset{(33)}{=} \sum \frac{\omega_n}{2}(p_n^2 + q_n^2)$$ (230) (188) (223)

$\Rightarrow$ Hamiltonian equations for p_n, q_n

$$\dot{q}_n = \frac{\partial}{\partial p_n} H = \omega_n p_n + \frac{\partial}{\partial p_n} H_{\text{Ⓘ}} \qquad (224)\ (190)$$

$$\frac{\partial}{\partial p_n} H_{\text{Ⓘ}} = \sum_j \Big(- \frac{q_j}{m_j} p_j \underbrace{\frac{\partial}{\partial p_n} A(r_j,t)}_{\text{Eq.}\overset{218}{(226)},(227)\ (185)\ \frac{1}{2i\sqrt{\varepsilon_0 \omega_n}}(i\vec{e}_n(r_j) + i e_n^*(r_j)) = \frac{Re[e_n^*(r_j}{\sqrt{\varepsilon_0\omega_n}}} + \sum_j \frac{q_j^2}{2m_j} \cdot 2\vec{A} \frac{\partial}{\partial p_n} \vec{A}(\vec{r}_j,t)$$

$$= -\sum_j q_j \underbrace{\left(\frac{P_j - q_j \vec{A}(\vec{r}_j,t)}{m} \right)}_{\text{Eq.}(222)\ \dot{\vec{r}}_j\ (180)} \cdot \frac{Re[\epsilon_n(r_j)]}{\sqrt{\varepsilon_0 \omega_n}} =$$

$$= -\frac{1}{\sqrt{\varepsilon_0 \omega_n}} Re\Big[\int dx^3\, e_n^*(x) \cdot \underbrace{\sum_j q_j \dot{\vec{r}}_j\, \delta(x - \vec{x}_j)}_{\text{Eq.}(213)\ j(x,t)\ (203)} \Big) \qquad (191)(225)\ (231)(226)$$

$$\Rightarrow \quad \dot{q}_n = \omega_n P_n - Re[j_n] \qquad (232)\ (172) \qquad (192)(227)$$

$$j_n = \int dx \frac{e_n^*(x)}{\sqrt{\varepsilon_0 \omega_n}} j(x,t) \qquad (233) \qquad (193)((228)$$

$\Rightarrow$ similarly $\quad \dot{P}_n = -\frac{\partial}{\partial q_n} H = -\omega_n q_n - Im[j_n] \qquad (234)\ (194)\ (229)$

$(192) \quad (194)$

Comparing Eqs (232)-(234) with Eqs. (32) and (24c)

one can see that Hamilton equations with H in Eq.(22

are equivalent to transversal Maxwell equations

where the current is given by Eq.(213b) (172)

Note: H describes classical dynamics of the system field + particles represented via pairs of canonical variables (independent): $\vec{r}_j, \vec{P}_j$ and q_n, P_n.

Quantization: $\vec{r}_j, \vec{P}_j, q_n, P_n \longrightarrow \hat{\vec{r}}_j, \hat{\vec{P}}_j, \hat{q}_n, \hat{P}_n$

$\hat{q}_n$ and $\hat{p}_n$ can be further represented by $\hat{a}_n$ and $\hat{a}_n^\dagger$

$$\hat{H} = \hat{H}_0 + \hat{H}_I + \hat{H}_F \qquad (235a) \qquad (195a) \qquad (230)$$

$$H_0 = \sum_j \frac{\hat{\vec p}_j^{\,2}}{2m_j} + V_{coul}(\hat{\vec r}_1, \ldots \hat{\vec r}_N) \qquad (235b)\ (195b)\ (240)$$

$$\hat H_I = \sum_j \left(-\frac{q_j}{m_j} \hat{\vec p}_j \cdot \hat{\vec A}(r_j) + \frac{q_j^2}{2m_j} \hat{\vec A}^2(r_j) \right) \qquad (235c)\ (195c)\ (241)$$

with $\hat{\vec A}(\vec r_j) = \hat{\vec A}^{(+)}(\vec r_j) + \hat{\vec A}^{(-)}(\vec r_j)$ (235d) (195d) (242)

$$\hat{\vec A}^{(+)}(\vec r_j) = \sum \sqrt{\hbar/2\epsilon_0\omega_n}\ \frac{1}{i}\ \hat a_n \vec e_n(\vec r_j) \qquad (235e)\ (195e)\ (243)$$

$$\hat{\vec A}^{(-)}(r_j) = \left(\hat{\vec A}^{(+)}(r_j)\right)^+ \qquad (235f)\ (195f)\ (244)$$

$$\hat H_F = \sum_n \hbar\omega_n(\hat a_n^+ \hat a_n + 1/2) \qquad (235g)\ (195g)\ (245)$$

Note: since $\mathrm{div}\, \vec e_n = 0$ $\quad \hat{\vec p}_j \cdot \hat{\vec A}(r_j) = \hat{\vec A} \cdot \hat{\vec p}_j$

- $\hat H_F$ can be expressed via $\hat{\vec E}$ and $\hat{\vec B}$ (51) (52)

$$\hat H_F = \int d^3x \left(\frac{\epsilon_0}{2} \hat{\vec E}^2(x) + \frac{1}{2\mu_0} \hat{\vec B}^2(x) \right) \qquad (246)\ (236)\ (196)$$

To prove this you substitute expressions for the field in terms of $\hat a$ and $\hat a^+$ into Eq. (236) (196) and obtain Eq. (235g). (195g)

States. Basis Fock space is formed as

$$R = R_{Matter} \otimes R_{Field}$$

R_{Matter} particle states without field, R_{field} - free field states.

VI.1 Exchange interaction ~~between~~ between ~~[illegible]~~ fields and ~~density in~~ the multipol ~~expansion~~ density

- Modell (235) can be formulated in many equivalent forms, for example via unitary transformation (247) (197a)

$$\hat U \hat U^+ = 1$$

=> If all operators are transformed as $\hat{\vec O} = \hat U^{-1} \hat O \hat U$ (237a) (197b)

and all states as $|\tilde\psi\rangle = \hat U^{-1} |\psi\rangle$ (237b) (197c)

Then this change in description does not change measured quantities.

$$H = H_0 + H_I + H_F \qquad (217)$$

$$H_0 = \sum_i \frac{\vec{p}_i^{\,2}}{2m_j} + V_{coul} \qquad V_{coul} = \frac{1}{2}\sum_{i \neq j} \frac{q_j q_i}{4\pi\varepsilon_0 |r_i - r_j|} \qquad (217a)$$

$$H_I = \sum_i \left[-\frac{q_i}{m_i} \vec{p}_i \cdot \vec{A}(\vec{r}_i, t) + \frac{q_j^2}{2m_j} \vec{A}^2(r_j, t) \right] \qquad (217b)$$

$$H_F = \int dx^3 \left(\frac{\varepsilon_0}{2} \underset{(!)}{\vec{E}_w^2} + \frac{1}{2\mu_0} \vec{B}^2 \right) \qquad (217c)$$

~~Quantization~~

$$A = \sum_n a_n(t)\, \vec{e}_n(r) \qquad (218)$$

$$\vec{B} = \mathrm{rot}\,\vec{A} = \sum_n a_n \underbrace{\mathrm{rot}\,\vec{e}_n}_{\omega_n / c \vec{b}_n} = \underbrace{\sum \sqrt{\mu_0 \omega_n}\, g_n \vec{B}_n}_{B}$$

$$\Rightarrow a_n = \frac{1}{\sqrt{\omega_n \mu_0}} g_n \qquad (219)$$

$$\vec{A} = \vec{A}^+ + \vec{A}^- \qquad A^{(-)} = \left(A^{(+)}\right)^\dagger$$

$$A^{(+)} = \sum_n \frac{1}{2i\sqrt{\varepsilon_0 \omega_n}} \left(q_n(t) + i p_n(t)\right) \vec{e}_n \qquad (220) \qquad \longrightarrow$$

$$H_F = \sum_n \frac{\omega_n}{2}\left(p_n^2 + q_n^2\right)$$

- Quantization $\underbrace{p_j, q_j}_{mat}, \underbrace{p_n, q_n}_{field} \longrightarrow \hat{p}_i, \hat{q}_i, \hat{p}_n, \hat{q}_n$

- States $|R\rangle = |R_{matter}\rangle \otimes |R_{field}\rangle$

Example: Power-Zierau-Wooley Transformation

$$\hat{u} = \exp\left[\frac{i}{\hbar}\int d^3\vec{r}\;\hat{\vec{P}}(r)\cdot\hat{\vec{A}}\right] \qquad (238)\ \overset{(247)}{(198)}$$

$$\hat{\vec{P}}(r) = \sum_j r_j q_i \int_0^1 \delta(r-\lambda r_j)\,d\lambda \qquad (239)\ \overset{(248)}{(199)}$$

$$\Rightarrow\quad \hat{u} = \exp\left[\frac{i}{\hbar}\sum_j q_j r_j\cdot\int_0^1 \vec{A}(\lambda r_j)\,d\lambda \ (240)\right] \quad (~~\#~~200).$$

(249)

$\vec{P}$ has the meaning of polarization

$$\operatorname{div}\vec{P} = -\rho(r) + \rho_0(r) \qquad (241)\ \overset{(250)}{(201a)}$$

with
$$\rho(r) = \sum_j q_j\,\delta(r-r_j) \qquad (242)\ (251)(201b)$$

$$\rho_0(r) = \sum_j q_j\,\delta(r) \qquad (243)\ \overset{(252)}{(201c)}$$

Transformation operators (237a) ~~together with~~ and (240) can be calculated (R. London, Quantum theory of Light, p. 166)

One has:

$$\hat{\vec{P}}_{j\,kin} = u^{-1}\hat{\vec{P}}_{j\,kin}\,u = u^{-1}(\hat{\vec{P}}_j - q_j\hat{\vec{A}}_j)\,\hat{u} = u^{-1}\cdot \qquad (253)$$

$$(\vec{P}_i - q_i\vec{A}(r_i))\,u = \hat{\vec{P}}_i + q_i\int_0^1 d\lambda\,\lambda[\vec{r}_j\times\hat{\vec{B}}(\lambda r_j)] \qquad \overset{(222)\,(180)}{} \quad \overset{(202a)}{(244a)}\ (254)$$

where $\hat{B}$ is the magnetic field operator. i.e. Eq(52). Further

$$\hat{\vec{B}}(r) = u^{-1}\hat{\vec{B}}\,u = \hat{\vec{B}} \qquad (244b)\ (202b)$$

(25C)

$$\hat{\vec{E}} = u^{-1}\hat{E}\,u = \hat{\vec{E}}_r - \frac{1}{\varepsilon}\hat{\vec{P}}_w(r) \qquad (244c)\ (202c)$$

(256)

$$\vec{P}_w(r)_e = \int d^3r'\sum_{e'}\delta^{\perp}_{ee'}(r,r')\,\vec{P}(r')_{e'} \qquad (244d)\ (202d)$$

(857)

Eq. (244d) is a traswersal part of the polarization

Note: In this representation the observable "electrical field" is defined by operator $\vec{E}(r,t)$. Operator with definition (51), defines ~~them~~ ~~more~~ the dielectrically shifted (dielectrische verschiebugngsfeld) (dielectric displacement) $D(r)$, so $\hat{D}(r) = \hat{\vec{E}}(r)$

- Finding eigenstates of $\hat{H}$ in Eq. (217) is very complicated. → theory of many-body systems (not quantum optics)

- Hamiltonian is not unique → any unitary transformation can be used to produce new form of H. $H' = U^{+} H U$.

VIII. Example: Power-Zierau-Wooley transformation.

$$U = \exp\left[\frac{i}{\hbar}\int d^3r\, \vec{P}(r)\cdot\vec{A}(r)\right] \qquad (222)$$

$$\vec{P} = \sum_j r_j q_j \int_0^1 \delta(r - \lambda r_j)\, d\lambda$$

$$\operatorname{div}\vec{P} = -\rho(r) + \rho_0(r)$$

$$\rho = \sum_j q_j\, \delta(r - r_j) \qquad \rho_0 = \sum_i q_i\, \delta(r)$$

Transformations (R. London, Quantum theory of light, p. 166)

$$\begin{cases} \left(\hat{\vec{P}}_j - \vec{q}_i \hat{A}(r_i)\right)' = U^{+}(\hat{p}_i - q_i\hat{A}_i)U = \hat{p}_i + q_i\int_0^1 d\lambda\, \lambda[\hat{r}_i \times \hat{B}(\\ \hat{\vec{B}}' = \hat{\vec{B}} \qquad\qquad (223) \\ \hat{\vec{E}}' = \hat{\vec{E}} - \frac{1}{\varepsilon_0}\hat{\vec{P}}_W \qquad (\hat{\vec{E}} \to \hat{\vec{D}} \text{ dielectric displacement}) \end{cases}$$

$$\hat{\vec{P}}_W = \int d^3r' \sum_{e'} \delta^{\perp}_{ee'}(r, r')\, \vec{P}(r')_{e'}$$ – projection onto W part

$$\Rightarrow \hat{H}' = U^{+}\hat{H}U = \hat{H}_0' + \hat{H}_I' + \hat{H}_F' \qquad (224)$$

$$\hat{H}_0' = \sum_j \frac{\hat{\vec{p}}_j^{\,2}}{2m_j} + V_{coul} + \underbrace{\frac{1}{2\varepsilon_0}\int d^3x\, \hat{\vec{P}}_W^{\,2}(x)}_{\text{is a function of } \vec{r}_j} \qquad (224a)$$

$$\hat{H}_I' = +\sum_j \frac{q_j}{2m_j}\int_0^1 d\lambda\,\lambda \Big\{\big(\hat{\vec{p}}_i \cdot [\hat{\vec{r}}_j \times \hat{\vec{B}}(\lambda r_j)]\big) + \big([\hat{\vec{r}}_j \times \vec{B}(\lambda r_j)]\cdot \hat{\vec{p}}_i\big)\Big\} +$$

$$+ \sum_j \frac{q_j^2}{2m_j}\left(\int_0^1 d\lambda\,\lambda\,[\hat{\vec{r}}_j \times \hat{\vec{B}}(\lambda r_j)]\right)^2 +$$

$$+ \int dx^3\, \vec{D}(x)\, \vec{P}(x)$$

$$H_I' = \hat{H}_{E-I} + \hat{H}_{M-I} + \hat{H}_{NL-I} \qquad (224b)$$

$$\hat{H}_F' = \sum_n \hbar\omega_n\left(\hat{a}_n^{+}\hat{a}_n + 1/2\right) \qquad (224c)$$

- Note: modes for the dielectric displacement D!

Note:

- Separation onto "material" and "field" variables is not unique.
- Advantages of PZW transformation →
  multipole expansion for small systems (atoms, nano-objects)

$E(r_j\lambda)$ and $B(\lambda r_j)$ are expanded using Taylor expansion

$$\hat{H}'_{E-I} = -\int d^3x\, \hat{\vec{D}}(x)\, \hat{\vec{P}}(x) \underset{\text{Eq. (222)}}{=} -\sum_i q_i \hat{\vec{r}}_j \int_0^1 d\lambda\, \hat{\vec{D}}(\lambda r_j) \approx$$

$$\approx -\sum_j q_j \hat{\vec{r}}_j \int_0^1 d\lambda\, \hat{\vec{D}}(0) = -\hat{\vec{P}}_0 \cdot \hat{\vec{D}}(0) \qquad (225)$$

$\hat{\vec{P}}_0 = \sum q_j \hat{\vec{r}}_j$ – dipole moment (operator)

$$\hat{H}'_{M-I} = \sum_j \frac{q_j}{4m_j}\left\{ \hat{\vec{p}}_j \cdot \left[\hat{\vec{r}}_j \times \hat{\vec{B}}_0(0)\right] + \left[\hat{\vec{r}}_j \times \hat{\vec{B}}(0)\right]\hat{\vec{p}}_j \right\} =$$

$$= -\sum_j \frac{q_j}{2m_j} \hat{\vec{L}}_j \cdot \hat{\vec{B}}(0) \qquad (226)$$

$2\hat{L}_j = [\hat{\vec{r}}_j \times \hat{\vec{p}}_j] - [\hat{\vec{p}}_j \times \hat{\vec{r}}_j]$ – angular momentum

⇒ one can introduce simplified models for interaction between fields and matter

Example: James-Cuming model.

Atom has two quantum levels (variables p_j, r_j)

$$\hat{H}'_0 = \hbar\omega_G |G\rangle\langle G| + \hbar\omega_A |A\rangle\langle A| \qquad (227)$$

→

Dipole interaction

$$\hat{\vec{P}}_0 = \vec{\mu}_0 |G\rangle\langle A| + \vec{\mu}_0^* |A\rangle\langle G|$$

$$\vec{\mu}_0 = \langle G| q\vec{r} |A\rangle$$

$D_0 = \sqrt{\frac{\hbar\omega_R}{2\varepsilon_0}}\, \vec{e}_R(0)(\hat{a} + \hat{a}^+)$ - single mode.

$$H_I' = \hbar g\, (\hat{a}^+ + \hat{a})\, |G\rangle\langle A| + c.c.$$

$$g = -\sqrt{\frac{\hbar\omega_R}{2\varepsilon_0}}\, \vec{\mu}_0\, \vec{e}_R(0)$$

$$\hat{H}_F' = \hbar\omega_R\, (a^+a + 1/2)$$

- J-C model is very popular in studies of the quatum optics of atoms/ small objects.

Exam: first week of August, 7 August

Then → in Oktober.

$\Rightarrow$ transformation of the Hamiltonian operator:

$$\bar{\hat{H}} = \hat{U}^{-1}\hat{H}\hat{U} = \sum_j \frac{1}{2m_j}\left(\hat{\vec{P}}_j + q_j\int_0^1 d\lambda\,\lambda[\vec{r}_j \times \hat{\vec{B}}(\lambda r_j)]\right)^2 +$$

$$+ V_{Coulomb}(r_j) + \int d^3x\,\frac{1}{2\varepsilon_0}\vec{P}_W^2(x) + \underbrace{\int d^3x\left(\frac{\varepsilon_0}{2}\hat{D}^2(x) + \frac{1}{2\mu_0}\hat{B}^2(x)\right)}$$

$$-\int d^3x\,\hat{\vec{D}}(x)\vec{P}(x) = \bar{\hat{H}}_0 + \bar{\hat{H}}_I + \bar{\hat{H}}_F \quad (245) \quad \xrightarrow{H_F}(203)$$

(258)

where
$$\bar{\hat{H}}_0 = \sum_j \frac{\bar{p}_j^2}{2m_j} + V_{coul}(r_1, \dots r_N) + \underbrace{\frac{1}{2\varepsilon_0}\int dx^3\,\hat{\vec{P}}_W^2(x)}_{} \quad (24 \quad (20$$

using Eqs. (239)(244d) (199)(202d) can be expressed via $\vec{r}_j$

(259)

$$\bar{\hat{H}}_I = -\underbrace{\int d^3x\,\hat{\vec{D}}(x)\cdot\vec{P}(x)}_{H_{E-I}} + \underbrace{\sum_j \frac{q_j}{2m_j}\int_0^1 d\lambda\lambda\left(\hat{\vec{P}}_j\cdot[\vec{r}_j \times \hat{\vec{B}}(\lambda r_j)] + ([\vec{r}_j \times B(\lambda r_j)])\hat{P}_j\right\}}_{H_{M-I}} +$$

$$+ \underbrace{\sum_j \frac{q_j^2}{2m_j}\left(\int_0^\lambda d\lambda\,\lambda[\vec{r}_j \times \hat{\vec{B}}(\lambda\vec{r}_j)]\right)^2}_{\tilde{H}_{NL-I}} \quad (246b)(204$$

(260)

$$\bar{\hat{H}}_F \overset{=H_F}{=} \sum_n{}' \hbar\omega_n(\hat{a}_n^+\hat{a}_n + 1/2) \quad (24 \quad (204$$

Note: $\bar{\hat{H}}_0 \neq \tilde{\hat{H}}_0 = \hat{U}^{-1}\hat{H}_0\hat{U}$ $\Rightarrow$ separation of total energy into "material" and "field" parts is not unique! (It depends on representation)

Advantages of this representation:

1) Instead of vector potential one uses fields E and B That are measurable quantities.

2) ~~The dipole density can be changed for the~~ The interaction can be written in the form of the multipole expansion.

This is useful in description of small objects such as atoms interacting with e-m radiation.

Exampee: atom (all charges are located in the vicinity of r=0). Field modes with optical range frequencies have ~~small~~ wavelengths $\lambda \gtrsim 100$n

$\Rightarrow$ the $\vec{e}_n(r)$ and $\vec{b}_n(r)$ change little on the scale of atoms ~ 1nm, and can be considered constant

$\Rightarrow$ Expansion representation for $\vec{E}(r_j\lambda)$ and $\vec{B}(r_j\lambda)$ for small $r_j\lambda \approx 0$ is meaningful

$$\Rightarrow \hat{H}_{E-W} = -\int^3 d^3x\, \hat{\vec{D}}(x)\cdot\hat{\vec{P}}(x) \underset{(239)}{=} -\sum_j q_j\vec{r}_j \int_0^1 d\lambda\, \hat{\vec{D}}(\lambda\vec{x}_j) \approx$$

$$\approx -\sum_j q_j\vec{r}_j \int_0^1 d\lambda\, \hat{\vec{D}}(0) = -\hat{\vec{P}}_0\cdot\hat{\vec{D}}(0) \quad (270) \quad (247) \quad \cancel{(205a)}$$

$$\hat{\vec{P}}_0 = \sum_j q_j\vec{r}_j \quad \text{dipole moment} \quad (248) \quad (205b)$$

- Next term in the expansion describe interation of the efield with quadrupole moment of the media charges field.

The other terms is interaction with the magnetic dipole

$$H_{M-I} = \sum_j \frac{q_j}{4m_j}\left\{\hat{\vec{P}}_j\cdot(\vec{r}_j\times\hat{\vec{B}}(r=0)) + (\vec{r}_j\times\hat{\vec{B}}(r=0))\cdot\vec{P}_j\right\} =$$

= change order by shift $-\sum_j \frac{q_j}{2m_j}\hat{\vec{L}}_j\cdot\vec{B}(0)$ $(2\cancel{8}1)$ (249) (206a)

$$\hat{\vec{L}}_j = \hat{r}_j\times\hat{p}_j \quad (206b)$$

Advantage of this representation

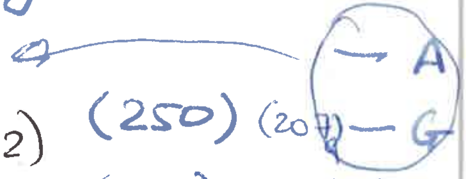
3) it is suitabee when the number of quantum levels (stats) in the material system is restricted.

VI. 2. The Jaynes - Cumming Model

Note: Atom in microresonator (e.g. Fabry-Perot)

=> The modes are descrete real standing waves

- Idealized modell: 2 level atom resonantly coupled with a single resonator mode.



$$\Rightarrow \hat{H}_0 = \hbar\omega_G |G\rangle\langle G| + \hbar\omega_A |A\rangle\langle A| \quad (272) \qquad (250)\ (207a)$$

$$\hat{P}_0 = \mu_0 |G\rangle\langle A| + c.c. \qquad (251)\ (207b)$$

$$\mu_0 = \langle G| q x |A\rangle \qquad (252)\ (207c)$$

$$\hat{\vec{D}}_0 = \sqrt{\frac{\hbar\omega_R}{2\varepsilon_0}}\, \vec{e}_R(0)(\hat{a} + \hat{a}^+) \qquad (253)\ (207d)$$

$\hat{a}$ - annihilation operator, $\vec{e}_R(r)$, ω_R - modes function, frequency

- Normalization $\vec{e}_R(0) \sim \frac{1}{\sqrt{V_R}}$ V_R - Volume of the resonator
- Only effective dipole coupling is taken into account as dominant

$$\Rightarrow \hat{H}_I = \hbar g(a^+ + a)|G\rangle\langle A| + c.c. \qquad (254)\ (208a)$$

$$\hbar g = -\sqrt{\frac{\hbar\omega_R}{2\varepsilon_0}}\, \vec{M}_0 \cdot \vec{e}_R(0) \qquad (255)\ (208b)$$

$$\hat{H}_F = \hbar\omega(a^+ a + 1/2) \qquad (256)\ (208d)$$

- $\hat{H}_I$ ~~is~~ consists of 4 contributions

$\sim \{\hat{a}^+ |G\rangle\langle A|$; $\sim \hat{a} |A\rangle\langle G|$; $\sim \hat{a}^+ |A\rangle\langle G|$ $\hat{a}|G\rangle\langle A|$

1. Photon created in transition $|A\rangle \to |G\rangle$ | 2. Photon annih. $|G\rangle \to |A\rangle$ | 3. Photon created $|G\rangle \to |A\rangle$ 4. Photon absor $|A\rangle \to$

- Both 3 and 4 processes are off-resonant
- We use rotating wave appoximation

$$\Rightarrow \hat{H}_{JC} = \hbar\omega_G |G\rangle\langle G| + \hbar\omega_A |A\rangle\langle A| + \hbar g \hat{a}^+ |G\rangle\langle A| + \hbar g^* \hat{a} |A\rangle\langle G| + \hbar\omega_R(\hat{a}^+\hat{a} + 1/2) \quad (273)\ 257)\ (209).$$

- The set of basis states

$$|G,n\rangle = |G\rangle \otimes |n\rangle \quad \text{and} \quad |A,n\rangle = |A\rangle \otimes |n\rangle \qquad n=0,1,2\dots$$

$\Rightarrow$ states

$$|\psi(t)\rangle = \sum_{n=0}^{+\infty} \left(\alpha_n(t)\,|G,n\rangle + \beta_n(t)\,|A,n\rangle\right) \qquad \frac{(274)}{(257)}\ (209)$$

Using: $\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ ~~(210)~~

$\Rightarrow$ Field equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = i\hbar\sum_{n=0}^{+\infty}\left(\dot{\alpha}_n|G,n\rangle + \dot{\beta}_n|A,n\rangle\right) = \hat{H}_{JC}|\psi\rangle =$$

$$= \hbar\omega_G\sum_n \alpha_n|G,n\rangle + \hbar\omega_A\sum_n \beta_n|A,n\rangle +$$

$$+ \hbar\omega_R\sum_n (n+1/2)\left(\alpha_n|G,n\rangle + \beta_n|A,n\rangle\right) + \qquad (275) \quad (210)$$

$$+ \hbar g\sum_n \sqrt{n+1}\,\beta_n|G,n+1\rangle + \sum_n \hbar g\sqrt{n}\,\alpha_n|A,n-1\rangle \qquad (258)$$

Projecting on $\langle n, G|$ 276 (211)

$$i\dot{\alpha}_n = \left(\omega_G + \omega_R(n+1/2)\right)\alpha_n + g\sqrt{n}\,\beta_{n-1} \qquad (259a)$$

Projecting on $\langle n-1, A|$ (212)

$$i\dot{\beta}_{n-1} = \left(\omega_A + \omega_R(n-1/2)\right)\beta_{n-1} + g^*\sqrt{n}\,\alpha_n \qquad (259b)$$

Important: α_n is only with β_{n-1} coupled $\Rightarrow$ Eq. (259) (213
can be solved for each n.

- α_0 is not coupled $\quad \alpha_0(t) = \alpha(t=0)e^{-i(\omega_G+\omega_R 1/2)t}$ (260)

the system in the ground state (without photons) remains in the ground state

$$\text{Substitution} \quad \overline{\omega_n} = \frac{\omega_A+\omega_G}{2} + n\omega_R \qquad (277)\ (214a)\ (261)$$

$$\delta\omega = \omega_A - \omega_G - \omega_R \qquad (262)\ (214b)$$

$$\omega_n + \frac{\delta\omega}{2} = \omega_A + (n-1/2)\omega_r$$

$$\omega_n - \delta\omega/2 = \omega_G + (n+1/2)\omega_r$$

$$\omega_A = \omega_G + \omega_R \rightarrow \omega_n = \frac{\omega_G + \omega_R}{}$$

und $\tilde{\alpha}_n = e^{i\omega_n t}\alpha_n$ (263a) (215a)

$\tilde{\beta}_{n-1} = e^{i\omega_n t}\beta_n$ (263b) (215b)

=> (259)
$$i\frac{\partial}{\partial t}\begin{pmatrix}\tilde{\alpha}_n\\ \tilde{\beta}_{n-1}\end{pmatrix} = \underbrace{\begin{pmatrix}-\delta\omega/2 & g\sqrt{n}\\ g^*\sqrt{n} & \frac{\delta\omega}{2}\end{pmatrix}}_{\tilde{U}_n}\begin{pmatrix}\tilde{\alpha}_n\\ \tilde{\beta}_{n-1}\end{pmatrix}$$ (278) (264) (216)

=> $\tilde{\alpha}_n$ und $\tilde{\beta}_{n-1}$ are calculated by finding eigenvalues of the matrix U.

$$|U_n-\lambda| = -\left(\frac{\delta\omega}{2}+\lambda\right)\left(\frac{\delta\omega}{2}-\lambda\right) - |g|^2 n = \lambda^2-\left(\frac{\delta\omega}{2}\right)^2-g^2 n = 0$$

=> $\lambda_\pm = \pm\frac{1}{2}\sqrt{\delta\omega^2+4|g|^2 n}$ (279) (217) (269)

Special case $\delta\omega = 0$

Set $\Omega_n = 2|g|\sqrt{n}$ (Rabi frequency) (218) (266)

eigenvector for $\lambda_+ = \frac{\Omega_n}{2}$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ +\frac{g^*}{|g|}\end{pmatrix}$ (219)

$\lambda_- = -\frac{\Omega_n}{2}$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\frac{g^*}{|g|}\end{pmatrix}$ (280)

=> Solution of Eq. (264)

$$\begin{pmatrix}\tilde{\alpha}_n\\ \tilde{\beta}_{n-1}\end{pmatrix} = \frac{c_n^+}{\sqrt{2}}\begin{pmatrix}1\\ g^*/|g|\end{pmatrix}e^{-i\frac{\Omega_n}{2}t} + \frac{c_n^-}{\sqrt{2}}\begin{pmatrix}1\\ -g^*/|g|\end{pmatrix}e^{i\frac{\Omega_n}{2}t}$$ (281) (267 (220)

Initial condition: at $t=0$ the system is $|A\rangle$

=> $\tilde{\alpha}_n(0)=0$ $\tilde{\beta}_{n-1}(0) = \tilde{\beta}^{(0)}_{n-1}$ at $|A\rangle$ => $c_n^+ = -c_n^-$ and $\sqrt{2}c_n^+\frac{g^*}{|g|} \overset{!}{=} \beta^{(0)}_{n-1}$ (221)

$$\Rightarrow \tilde{\beta}_{n-1}(t) = \bar{\beta}^{(0)}_{n-1}\underbrace{\cos\left(\frac{\Omega_n}{2}t\right) \quad \tilde{\alpha}_n(t)}_{(282)} = -i\frac{|g|}{g^*}\tilde{\beta}^{(0)}_n\sin\left(\frac{\Omega_n}{2}t\right)$$

Inversion:

$$I(t) = \langle\psi|\left(|A\rangle\langle A| - |G\rangle\langle G|\right)|\psi\rangle = \sum_{n=0}^{+\infty}\left\{|\hat{\beta}_n(t)|^2 - |\alpha_{n+1}(t)|^2\right\} =$$

$$= \sum_{n=0}^{+\infty}|\hat{\beta}_n^{(0)}|^2\left(\underbrace{\cos^2\left(\frac{\Omega_{n+1}}{2}t\right) - \sin^2\left(\frac{\Omega_{n+1}}{2}t\right)}_{\cos(\Omega_{n+1}t) = \cos(2|g|\sqrt{n+1}\,t)}\right)$$ (283) (268) (222)

- If the initial state contains n-photons

$\rightarrow$ $I(t)$ oscillates with $\Omega_{n+1} = 2|g|\sqrt{n+1}$ frequency

Note: if the is no photons initially $(n=0)$

$I(t)$ oscillates with Vacuum Rabi frequency

(Vacuum Rabi oscillations $\Omega_0 = 2g$)

- If the initial state is coherent

$\Rightarrow$ (119)(120)

$$|\beta_n^{(0)}|^2 = e^{-\bar{n}} \frac{\bar{n}^n}{n!} \qquad (268) \quad (222)$$

$\Rightarrow$ (268)

$$I(t) = e^{-\bar{n}} \sum_{n=0}^{+\infty} \frac{\bar{n}^n}{n!} \cos\left(2\pi\sqrt{\frac{n+1}{\bar{n}+1}}\, t/\mathcal{T}_R\right) \qquad (269) \quad (223)$$

$$\mathcal{T}_R = \frac{2\pi}{\Omega_{\bar{n}+1}} = \frac{\pi}{|g|\sqrt{\bar{n}+1}} \qquad (284).$$

$I(t)$ periodically decays (collapses) and revives

- Collaps is due to destructive interference of incommensurate frequencies
- Revival appears ~~when~~ because frequencies Eq. (269) are discrete

dominant frequency: $\Omega_R = \Omega_{\bar{n}+1} = 2|g|\sqrt{n+1}$

neighbor frequency: $\Omega_{\bar{n}}$ has finite interval (next)

$\Rightarrow$ oscillations with Ω_R and $\Omega_{\bar{n}}$ interfere constructively at $t_{Rev}\left[\Omega_R - \Omega_{\bar{n}}\right] = 2\pi j \quad (j = 0, 1, 2..)$

$$\Rightarrow t_{Rev} = \frac{2\pi j}{2|g|\sqrt{\bar{n}+1} - \sqrt{\bar{n}}} = \frac{2\pi j}{2|g|\sqrt{\bar{n}}\underbrace{\left(\sqrt{1+1/\bar{n}} - 1\right)}_{\approx 1/2\bar{n}}} \approx \frac{2\pi}{|g|} j\sqrt{\bar{n}} \quad (2$$

$$\Rightarrow \frac{t_{Rev}}{\mathcal{T}_R} \underset{\bar{n} \gg 1}{\approx} 2\sqrt{\bar{n}(\bar{n}+1)} \approx 2\bar{n} \qquad (285) \quad (224) \qquad (271)$$

- Eigenstates in JC model:

$|G,0\rangle$ is an eigenstate with energy

$$E_0 = \hbar\omega_G + \hbar\omega_R\frac{1}{2} \qquad (271)$$

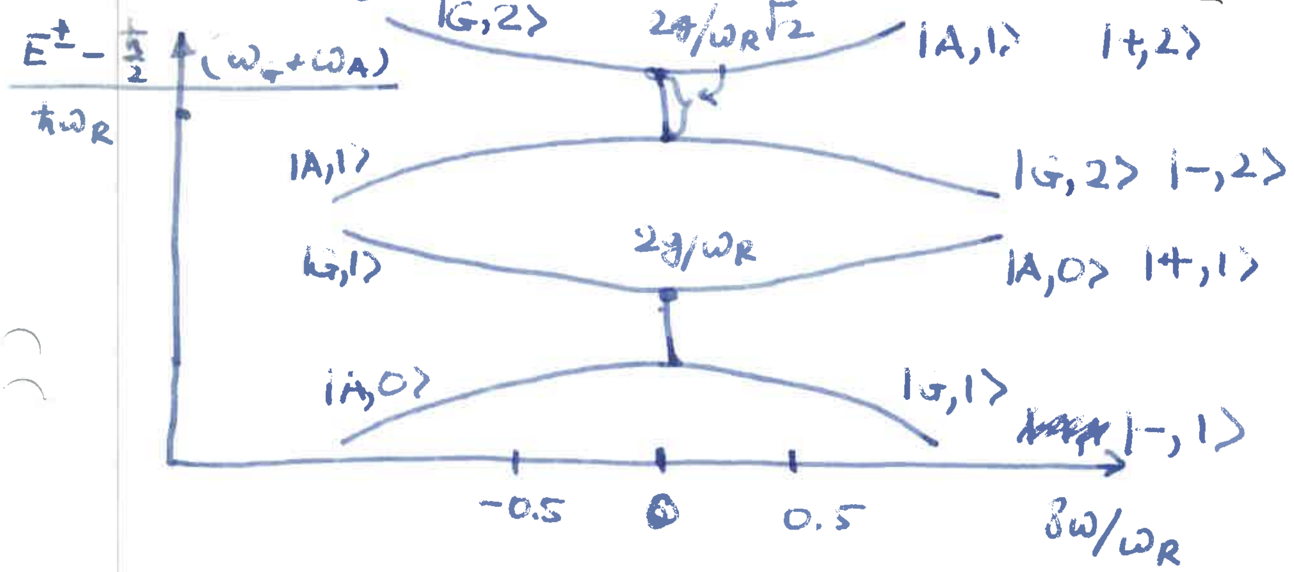
- For other states H_{JC} is represented in a block-diagonal form

$$\hat{H}_{JC}\begin{pmatrix}|G\,n\rangle\\|A\,n-1\rangle\end{pmatrix} = \hbar\begin{pmatrix}\omega_G+(n+1/2)\omega_R & g\sqrt{n}\\ g^*\sqrt{n} & \omega_A+(n-\frac{1}{2})\omega_R\end{pmatrix}\begin{pmatrix}|G\,n\rangle\\|A\,n-1\rangle\end{pmatrix}$$

$$\underset{(261)(262)(264)}{=} \hbar\bar{\omega}_n\begin{pmatrix}|G,n\rangle\\|A,n-1\rangle\end{pmatrix} + \hbar\,\hat{u}_n\begin{pmatrix}|G,n\rangle\\|A,n-1\rangle\end{pmatrix} \qquad (272)$$

⇒ eigenvalues (energy)

$$E_n^\pm = \frac{\hbar}{2}(\omega_G+\omega_A+2n\omega_R)\pm\sqrt{\delta\omega^2+4|g|n}\;\frac{1}{2} \qquad (273)$$



Energy gap at $\delta\omega = 0$ (Gap between normal modes)

$$E_n^+ - E_n^- = 2\hbar g\sqrt{n}$$

- When $n=1$ the state $|A,0\rangle$ (without photons) mixes with $|G,1\rangle$

The gap $E_1^+ - E_1^- = 2\hbar g$ is called Vacuum Rabi-gap.

Eigen values / eigenstates

$$\lambda_+ = \omega_n + \frac{\Omega_n}{2} \qquad \vec{\xi}_+ = \frac{1}{\sqrt{2\Omega_n}} \begin{pmatrix} \frac{2g\sqrt{n}}{\sqrt{\Omega_n+\delta\omega}} \\ \sqrt{\Omega_n+\delta\omega} \end{pmatrix}$$

$$\lambda_- = \omega_n - \frac{\Omega_n}{2} \qquad \vec{\xi}_- = \frac{1}{\sqrt{2\Omega_n}} \begin{pmatrix} \frac{2g\sqrt{n}}{\sqrt{\Omega_n-\delta\omega}} \\ -\sqrt{\Omega_n-\delta\omega} \end{pmatrix}$$

$$\Omega_n = \sqrt{\delta\omega^2 + 4g^2n}$$

$$\lambda_+(\delta\omega) = \omega_n + \sqrt{\delta\omega^2+4g^2n}$$

$$\lambda_+(\delta\omega \to +\infty) \approx \omega_n + |\delta\omega| + \frac{4g^2n}{|\delta\omega|}$$

$$\lambda_-(\delta\omega \to \infty) \approx \omega_n - |\delta\omega| - \frac{4g^2n}{|\delta\omega|}.$$

1. $\omega \to +\infty$

$$\vec{\xi}_+ \approx \frac{1}{\sqrt{2\delta\omega}} \begin{pmatrix} \frac{2g\sqrt{n}}{\sqrt{2\delta\omega}} \\ \sqrt{2\delta\omega} \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |A, n\rangle \qquad \Omega_n+\delta\omega \simeq 2\delta\omega$$

$$\vec{\xi}_- \approx \frac{1}{\sqrt{2\delta\omega}} \begin{pmatrix} \frac{2g\sqrt{n}\cdot\delta\omega}{\sqrt{2g\sqrt{n}}} \\ -\frac{\sqrt{2g\sqrt{n}}}{\sqrt{\delta\omega}} \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Omega_n-\delta\omega \simeq \frac{2g^2n}{\delta\omega}$$

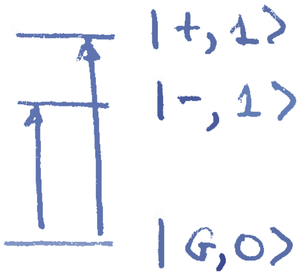
$|G, n\rangle$

2. $\omega \to -\infty$

$$\vec{\xi}_+ \approx \frac{1}{\sqrt{2\delta\omega}} \begin{pmatrix} \frac{2g\sqrt{n}}{\sqrt{2g\sqrt{n}}}\delta\omega \\ +\frac{\sqrt{2ng}}{\sqrt{\delta\omega}} \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |G, n\rangle$$

$$\vec{\xi}_- \approx \frac{1}{\sqrt{2\delta\omega}} \begin{pmatrix} \quad \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad |A, n-1\rangle$$

- Observations via optical excitations



VI. 3. Influence of losses.

- Becase mirrors are not perfect and similar influences photons can be lost from the volume

$\Rightarrow$ System = atom + resonator modes becomes an open system

then the meaningfull description is by using the density matrix.

Example: JC modell ~~is~~ in block diagonal form when ~~is~~ represented by the states : $|\nu\rangle \in |G,0\rangle, |G,1\rangle, |A,0\rangle$

$\nu = 0, 1, 2 \dots$

we assume $g = g^*$

$$\Rightarrow \hat{H}_{JC} = \sum_{\nu} \hbar \omega_\nu |\nu\rangle\langle\nu| + \hbar g \left(|2\rangle\langle 1| + |1\rangle\langle 2| \right) \quad (274)$$

Assume resonance case $\omega_R = \omega_A - \omega_G$

$$\Rightarrow \omega_1 = \omega_2 = \omega_G + 3/2\, \omega_R \quad (275)$$

Definition: $\hat{\rho}_{\nu\nu'} = |\nu\rangle\langle\nu'|$ (276)

$$\rho_{\nu\nu'} = \langle \hat{\rho}_{\nu\nu'} \rangle = Sp\left(\hat{\rho}_s \hat{\rho}_{\nu\nu} \right) = \langle \nu' | \hat{\rho}_s | \nu \rangle \quad (277)$$

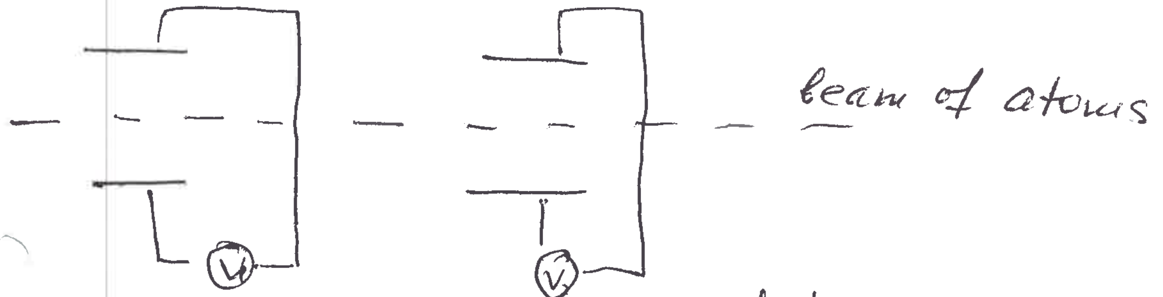
where $\hat{\rho}_s$ is a statistical operator

$$\Rightarrow \hat{\rho}_s = \sum_{\beta_j} \beta\, P_{\beta_j} |\beta_j\rangle\langle\beta_j| \quad (278)$$

$|\beta_j\rangle$ – orthogonal normalized basis.

$$\sum_{\beta_j} P_{\beta_j} = 1 \qquad 0 \leq P_{\beta_j} \leq 1 \quad (279)$$

- They can be easily ionised by aplying static field.
- This can be use for selective measurement of the occupation in the state $|n\rangle - |A\rangle$ and State $|n-1\rangle \rightarrow |G\rangle$



$V_1 < V_2$ detect occupation in $|G\rangle$

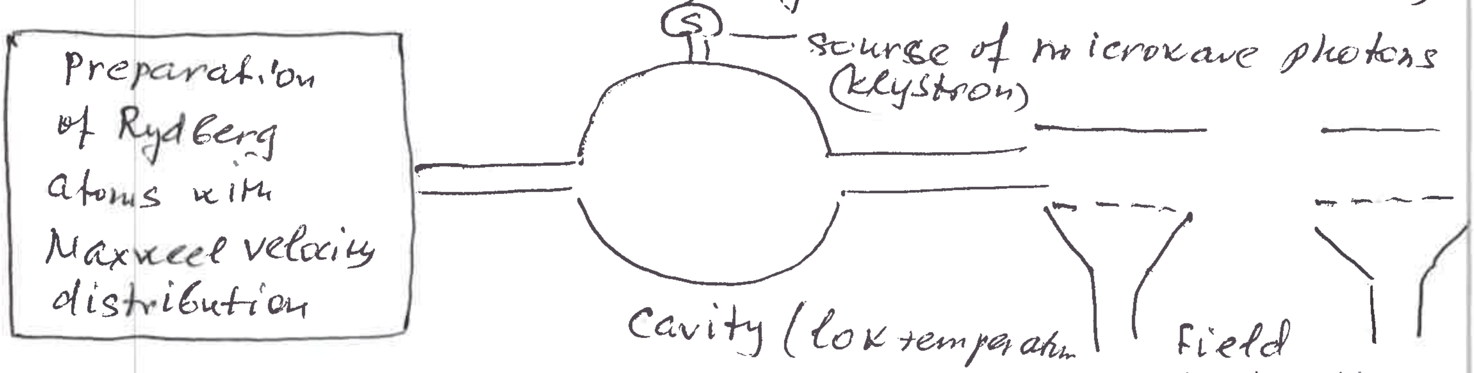
&

detects occupation in $|A\rangle$

- The ~~system~~ atomic states $|n\rangle$ ~~+~~$|A$ and $|n-1\rangle$ is a good approximation of a twolevel system, coupled to a microwave radiation $\lambda \sim$ mm.

VI.4.1. Experimental measurements of Rabi oscillatio

Haroche (Paris) Phy. Rev. Lett 76 1800 (1996)



- Detectors measure quantity of atoms on each levels
- For each detected atom there is a certain time of flight wich defines ~~the~~ ~~A~~ interation time and we measure the difference $\rho_{11} - \rho_{22}$
- Results: Rabi oscillations with frequency $g\sqrt{n}$
- Remark: interaction time can be ajusted by aplying an additional E-field (Stark shift → controlling Atom-Cavit

VI.4.2. Production of entangled atoms using CQED

Atom 1 flies first through the cavity with interaction time $t_I = \frac{1}{|g|}\pi/4$, ~~and~~ It is in ~~will acquire~~ an $|A\rangle_1$ state while the cavity is in $|0\rangle$ state. Its final state $(g = |g|)$

$$\overset{=}{\scriptstyle(267)} \quad |\Psi(t_I)\rangle = e^{-i\bar\omega_1 t_I}\left(-i\sin(gt_I)|G_1\rangle_1|1\rangle + \cos(gt_I)|A\rangle_1|0\rangle\right)$$

$$= \frac{e^{-i\bar\omega_1 t_I}}{\sqrt2}\left(-i|G\rangle_1|1\rangle + |A\rangle_1|0\rangle\right)$$

Atom 2 flies in second in a state $|G\rangle_2$.

$\Rightarrow$ Total ~~final~~ state $|\bar\Psi(t_I)\rangle = (-i|G\rangle_1|G\rangle_2|1\rangle + |A\rangle_1|G\rangle_2|0\rangle)$

Atom 2 interacts with cavity during $\Delta t = \bar t_I = \frac{1}{g}\pi/2$

$\Rightarrow |G_2\rangle|1\rangle \rightarrow \cos(g\bar t_I)|G\rangle_2|1\rangle - i\sin(g\bar t_I)|A\rangle_2|0\rangle = -i|A\rangle_2|0\rangle$

The state $|G\rangle_2|0\rangle$ does not change

Total state after interaction with atom 2 finishes

$$|\Psi\rangle_{end} = e^{-i\bar\omega_1(t_I+\bar t_I)}\frac{1}{\sqrt2}\underbrace{\left(-|G\rangle_1|A\rangle_2 + |A\rangle_1|G\rangle_2\right)}_{\text{entangled states}}|0\rangle$$

Coincidence measurement of states $\rightarrow$ PRL 79, 1 (1997) Haroche et al.

ygonal representation

define $\vec{\beta}_\nu = \begin{pmatrix} \sqrt{p_1} \langle \nu | \beta_1 \rangle \\ \sqrt{p_2} \langle \nu | \beta_2 \rangle \\ \vdots \end{pmatrix}$ (280)

$\Rightarrow \vec{\beta}_\nu^{\,*} \cdot \vec{\beta}_{\nu'} = \sum_{\beta_j}' p_{\beta_j} \langle \nu' | \beta_j \rangle \langle \beta_j | \nu \rangle \underset{(278)(277)}{=} \rho_{\nu\nu'}$ (281)

Important relation:

$|\rho_{\nu\nu'}|^2 = |\vec{\beta}_\nu^{\,*} \cdot \vec{\beta}_{\nu'}|^2 \underset{\text{Schwartzinequality}}{\leq} |\vec{\beta}_\nu|^2 |\vec{\beta}_{\nu'}|^2 = \rho_{\nu\nu} \cdot \rho_{\nu'\nu'}$ (282)

Equation of motion for the statistical operator

$\dot{\rho}_{\nu\nu'} = \frac{1}{i\hbar} [\hat{\rho}_{\nu\nu}, \hat{H}_{JC}] +$ relaxation terms (coupling with the environment) (283)

- Simple relaxation modell occupation of the $|G,1\rangle$ state, ρ_{11}, relaxes to the occupation of the $|G,0\rangle$ state, ρ_{00} (phonon leaves the system).
- Eq.(282) enforces the balance between the rates of diminishing the occupation terms: if the process of diminishing ρ_{11} is exponential $\rho_{11} \sim \exp[-t/\tau]$ then ρ_{01} and ρ_{12} are also exponential $\rho_{01} \sim \rho_{12} \sim \exp[-t/\tau_1]$ $\tau_1 = 2\tau$

$\Rightarrow$ use Eq. (283) to obtain equations for $\rho_{11}, \rho_{12}, \rho_{22}$, $[\rho_{00} = 1 - \rho_{11}, \rho_{01} = \rho_{10} = 0]$ $\rho_{12} = \rho_{21}^*$ $\Rightarrow$ $Y_{12} = \rho_{12} - \rho_{21}$

$$\begin{pmatrix} \dot{\rho}_{22} \\ \dot{\rho}_{11} \\ \dot{Y}_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 & ig \\ 0 & -1/\tau & -ig \\ 2ig & -2ig & -1/2\tau \end{pmatrix} \begin{pmatrix} \rho_{22} \\ \rho_{11} \\ Y_{12} \end{pmatrix} \quad (284)$$

eigenvalues of this matrix $(\lambda + 1/2\tau)(\lambda^2 + \lambda/\tau + 4g^2) = 0$

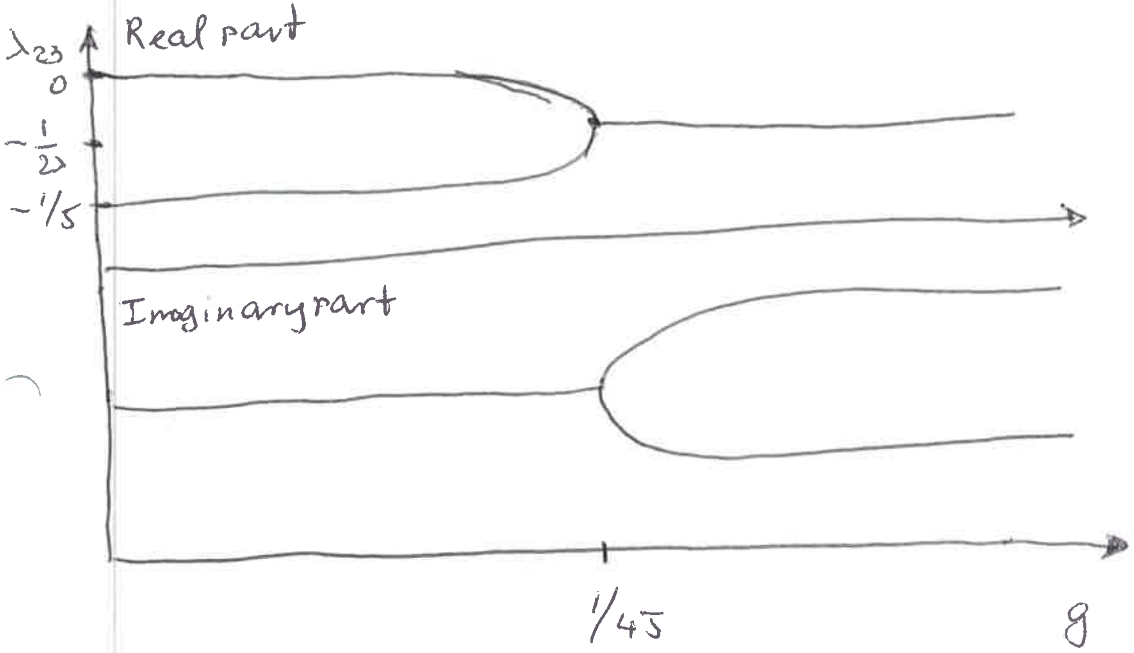
$\lambda_1 = -1/2\tau$ (286a) $\lambda_{2,3} = -1/2\tau \pm \sqrt{1/4\tau^2 - 4|g|^2}$ (286b)

- For $4g^2 < 1/4\Im^2$ all eigenstates are real

$\Rightarrow$ no Rabi oscillations only decay

- For $4g^2 > 1/4\Im^2$ there are two complex eigenstates

$\Rightarrow$ damped rabi ~~o~~oscillations (vacuum)



IV. 4. Experiment ~~with~~ on cavity – QED interaction with Rydberg Atom

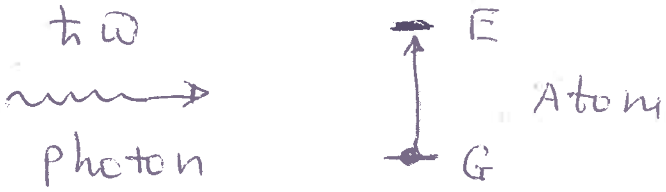
- Rydberg atom is atom with one electron placed in a state with high quantum number $n \gtrsim 50 - 733$
- The state for experiment has angular momentum $\ell = n-1$ $|m| = \ell$ => circular Rydberg atomic state
- ~~The~~ Rydberg atoms in circula states have
  - much larger dipole momentum because of much larger occupied space $d \sim n^2 q a_0$
  - long life time $\sim 1/10$ sec

Измерение света.

①

Рассмотрим, как происходит измерение фотона посредством взаимодействия фотона с атомом (материей). Общая схема измерения:

Фотон поглощается атомом, в результате чего атом меняет состояние: G (основное состояние) переходит в состояние E (возбужденное состояние). Измерение = определение нового состояния атома



⇒ Цель вычисления: рассчитать вероятность изменения состояния атома, из G в любое состояние E (≠ G).

⇒ Алгоритм вычисления: решение уравнения Шредингера

⇒ Модель (Гамильтониан)

$$\hat{H} = \hat{H}_M + \hat{H}_F + \hat{H}_I$$

$$\hat{H}_M = \hbar\omega_G |G\rangle\langle G| + \hbar\sum_E \omega_E |E\rangle\langle E|$$ – гамильтониан атома

$$\hat{H}_F = \hbar\sum_n \omega_n\left(\hat{a}_n^+\hat{a}_n + \frac{1}{2}\right)$$ – гамильтониан поля

$$\hat{H}_I = -\hat{P}\cdot\hat{D} = \underbrace{-q\cdot r}_{\text{дипольный момент атома}}\cdot\hat{D}(r)\,(=\hat{E}(r))$$ ← оператор амплитуд

=> Уравнение Шредингера

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$ + начальные условия

$$|\psi_{t=0}\rangle = |i\rangle \otimes |G\rangle$$

начальное состояние света

начальное состояние атома

=> Представление взаимодействия

$$\hat{H} = \hat{H}_0 + \hat{H}_I \qquad \hat{H}_0 = \hat{H}_M + \hat{H}_F$$

$$|\psi\rangle = e^{-i\hat{H}_0 t/\hbar}|\varphi\rangle$$

$$i\hbar\frac{d}{dt}\left[e^{-i\frac{\hat{H}_0 t}{\hbar}}|\varphi\rangle\right] = \hat{H}_0 e^{-i\frac{\hat{H}_0 t}{\hbar}}|\varphi\rangle + e^{-\frac{i\hat{H}_0 t}{\hbar}}\, i\hbar\frac{d}{dt}|\varphi$$

$$= (\hat{H}_0 + \hat{H}_I)\, e^{-\frac{i\hat{H}_0 t}{\hbar}}|\varphi\rangle$$

$$\Rightarrow \boxed{i\hbar\frac{d}{dt}|\varphi\rangle = \hat{\tilde{H}}_I(t)|\varphi\rangle \qquad \hat{\tilde{H}}_I(t) = e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{H}_I e^{-\frac{i\hat{H}_0 t}{\hbar}}}$$

Гамильтониан взаимодействия

$$\hat{\tilde{H}}_I(t) = -\hat{\tilde{P}}(t)\cdot\hat{\tilde{E}}(t) = e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{H}_I e^{-\frac{iH_0 t}{\hbar}} =$$

$$= -\underbrace{e^{\frac{i\hat{H}_M t}{\hbar}}(+q\cdot r)\,e^{-\frac{i\hat{H}_M t}{\hbar}}}_{\hat{\tilde{P}}(t)} \cdot \underbrace{e^{\frac{i\hat{H}_F t}{\hbar}}\,\hat{E}(r)\,e^{-\frac{iH_F t}{\hbar}}}_{\hat{\tilde{E}}(t)}$$

=> Решение уравнение Шредингера методом теории возмущений

$\hat{\tilde{H}}_I \Rightarrow \lambda \hat{\tilde{H}}_I$ где λ – формально маленькая величина

сделаем разложение решения в ряд Тэйлора по маленькой λ

$$|\varphi\rangle = |\varphi_0\rangle + \lambda|\varphi_1\rangle + \lambda^2|\varphi_2\rangle + \dots$$

$$i\hbar\frac{d}{dt}\left[|\varphi_0\rangle + \lambda|\varphi_1\rangle + \lambda^2|\varphi_2\rangle + \dots\right] =$$

$$= \hat{\tilde{H}}_I\left[|\varphi_0\rangle + \lambda|\varphi_1\rangle + \lambda^2|\varphi_2\rangle + \dots\right]$$

0: λ^0 — $i\hbar\frac{d}{dt}|\varphi_0\rangle = 0 \Rightarrow |\varphi_0\rangle$ - начальное состояние

1: λ^1 — $i\hbar\frac{d}{dt}|\varphi_1\rangle = \hat{\tilde{H}}_I(t)|\varphi_0\rangle$

2: λ^2 — $i\hbar\frac{d}{dt}|\varphi_2\rangle = \hat{\tilde{H}}_I(t)|\varphi_1\rangle$

⋮

$$\Rightarrow |\varphi_1\rangle = -\frac{i}{\hbar}\int_{+t_0}^{t}\hat{\tilde{H}}_I(\tau)|\varphi_0\rangle d\tau = -\frac{i}{\hbar}\int_{t_0}^{t}\hat{\tilde{H}}_I(\tau)d\tau\,|\varphi_0$$

$$\Rightarrow \boxed{|\varphi\rangle = |\varphi_0\rangle - \frac{i}{\hbar}\underset{(\equiv 1)}{\lambda}\int_{t_0}^{t}\hat{\tilde{H}}_I(\tau)d\tau\,|\varphi_0\rangle + \dots}$$

Начальные условия

$$|\psi_0\rangle = e^{-\frac{i\hat{H}_0 t}{\hbar}}\underbrace{|i,G\rangle}_{|\varphi_0\rangle}$$

=> Конечное состояние системы

$$|\Phi_f\rangle = \sum_E \sum_f \alpha_{Ef} \, |f,E\rangle$$

Коэффициент перехода в состояние $|E,f\rangle$

$$\alpha_{Ef} = \langle f,E|\Phi\rangle = \underset{=0}{\langle f,E|\Phi_0\rangle} - \frac{i}{\hbar}\int_{t_0}^{t} \langle f,E|\hat{\tilde{H}}_I|\Phi_0\rangle d\tau$$

$$\langle f,E|i,G\rangle = 0$$

Вероятность перехода $G,i \rightarrow E,f$

$$W_{fE} = |\alpha_{fE}|^2$$

Нужно посчитать матричный элемент

$$\langle f,E|\hat{\tilde{H}}_I|i,G\rangle = -\langle E|\underbrace{\hat{\tilde{\vec{P}}}}_{q\vec{r}}|G\rangle \cdot \langle f|\hat{\tilde{E}}|i\rangle$$

a) $\langle E|q\cdot\tilde{r}|G\rangle =$

$$= q\underbrace{\langle E|e^{+\frac{i\hat{H}_M t}{\hbar}}}_{e^{i\omega_E t}\langle E|} \, r \, \underbrace{e^{-\frac{i\hat{H}_M t}{\hbar}}|G\rangle}_{e^{-i\omega_G t}|G\rangle} = e^{i(\omega_E-\omega_G)t}\cdot\vec{P}_{EG}$$

$$\vec{P}_{EG} = q\langle E|r|G\rangle \equiv q\int dr^3 \, \varphi_E^*(r)\cdot r\,\varphi_G(r)$$

матричный элемент дипольного момента

b) $\langle f | \hat{\tilde{E}} | i \rangle$

$$\hat E = \hat E^{(+)} + \hat E^{(-)} \qquad \hat E^{(+)} = \sum_n \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}}\, \underbrace{\vec e_n(r)}_{\simeq \vec e_n(0)}\, \hat a_n$$

$$\hat E^{(-)} = \left(\hat E^{(+)}\right)^+$$

$$\hat{\tilde a}(t) = e^{\frac{i\hat H_F t}{\hbar}}\, \hat a_n\, e^{-\frac{i\hat H_F t}{\hbar}} = \hat a\, e^{-i\omega_n t} = \hat{\tilde a}(t)$$

Замечание: это соотношение следует из уравнений Гейзенберга, но можно получить и просто рассматривая матричные элементы

Тогда: $\hat{\tilde E}(t) = \hat{\tilde E}^{(+)}(t) + \hat{\tilde E}^{(-)}$

$$\hat{\tilde E}^{(+)}(t) = \sum_n \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}}\, \vec e_n(0)\, \hat a_n e^{-i\omega_n t}$$

$$\hat{\tilde E}^{(-)}(t) = \left[\hat{\tilde E}^{(+)}(t)\right]^+$$

$$\Rightarrow \langle f | \hat{\tilde E}(t) | i \rangle = \sum_n{}' \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}} \Big[\vec e_n(0) e^{-i\omega_n t} \langle f | \hat a_n | i \rangle$$

$$+ \vec e_n^{\,*}(0) e^{i\omega_n t} \langle f | \hat a_n^+ | i \rangle$$

=> Итак, амплитуда перехода

$$\alpha_{fE} = \frac{i}{\hbar}\int_{t_0}^{t} \vec{P}_{EG}\, e^{i(\omega_E-\omega_G)t} \cdot \sum_n' \sqrt{\frac{\hbar\omega_n}{2\varepsilon_0}} \times$$

$$\times\left\{ \langle f|\hat{a}_n|i\rangle e^{-i\omega_n \tau}\vec{e}_n(0) + \langle f|\hat{a}_n^+|i\rangle e^{i\omega_n \tau} \right\}$$

У нас есть 2 типа интегралов

1) $\int_{t_0}^{t} e^{i(\omega_E-\omega_G-\omega_n)\tau}d\tau$ – соответствует поглощению фотон
($\hat{a}_n$)

2) $\int_{t_0}^{t} e^{i(\omega_E-\omega_G+\omega_n)\tau}d\tau$ – соответствует испусканию фото
($\hat{a}_n^+$)

Теперь предположим, что измерения идут долго

"долго" означает, что характерное время ожидания (измерения) пока состояние ~~нам~~ атома изменится больше, чем все другие времена в системе. (~~изменение динамики атома~~)

Тогда $t_0 \to -\infty$ $t \to +\infty$

$$\to \int_{-\infty}^{+\infty} e^{i\omega\tau}d\tau = 2\pi\delta(\omega)$$

1)
2) $\int_{-\infty}^{+\infty} e^{i(\omega_E-\omega_G\mp\omega_n)\tau}d\tau = 2\pi\delta(\omega_E-\omega_G\mp\omega_n)$

Однако, нам не интересно конечное состояние фотона $|f\rangle$, нам важно, что атом перешел в состояние $|E\rangle$. Поэтому, мы просуммируем результат по состояниям $|f\rangle$

$$W_E = \sum_f W_{Ef} = \frac{1}{\hbar^2}\sum_{\alpha,\beta} p^{\alpha}_{EG}\, p^{\beta *}_{EG} \int_{t_0}^{t} d\tau' \int_{t_0}^{t} d\tau'' \times$$

$$\times e^{i\Delta_{EG}(\tau'-\tau'')} \sum_f \langle i|\, \hat{\tilde{E}}^{(-)}(\tau'')\, \underbrace{|f\rangle\langle f|}\, \hat{\tilde{E}}^{(+)}(\tau')|i\rangle$$

$$\sum_f |f\rangle\langle f| = \hat{\mathbb{1}}$$

(потому что система-бази состояний полная)

$$\Rightarrow W_E = \int_{t_0}^{t} d\tau' \int_{t_0}^{t} d\tau'' \sum_{\alpha\beta} K^{EG}_{\alpha\beta}(\tau'-\tau'')\; G^{(1)}_{\alpha\beta}(\tau',\tau'')$$

$$K^{EG}_{\alpha\beta}(\tau) = \frac{1}{\hbar^2}\, p^{\alpha}_{EG}\, p^{\beta *}_{EG}\, e^{+i\Delta_{EG}\tau}$$

$$G^{(1)}_{\alpha\beta}(\tau',\tau'') = \langle i|\, \hat{\tilde{E}}^{(-)}_{\beta}(\tau')\, \hat{\tilde{E}}^{(+)}_{\alpha}(\tau'')\,|f\rangle =$$

$$= \sum_{n,m} \frac{\hbar}{2\varepsilon}\sqrt{\omega_n\omega_m}\,\left[\vec{e}_n(0)^*\right]_{\beta}\left[\vec{e}_m(0)\right]_{\alpha} \langle i|\, a_n^{+} a_m |i\rangle \times$$

$$\times e^{-i\omega_m\tau' + i\omega_n\tau''}$$

корреляционная функция первого порядка

Нам даже не нужно знать, в каком именно состоянии находится атом, достаточно лишь знать, что он больше не в основном состоянии G. Тогда полная вероятность что он не в состоянии G дается суммой

$$P = \sum_E W_E \cdot R_E$$ (R_E – какая-то весовая функция, зависит от измерения состояния атома)

$$\Rightarrow P = \sum_{\alpha\beta}{}' \int_{t_0}^{t} dT' \int_{t_0}^{t} dT'' \int_{-\infty}^{+\infty} d\omega\, S_{\alpha\beta}(\omega)\, e^{i\omega(T'-T'')} \times G^{(1)}_{\alpha\beta}(T',T$$

$$S(\omega) = \frac{1}{\hbar^2} \sum_E{}' R_E \cdot P^{\alpha}_{EG}\, P^{\beta *}_{EG}\, \delta(\omega - \Delta_{EG}) \; -$$

– функция измерителя (атома + прибора, меря ющего его состояние)

$S(\omega)$ – зависит от частоты и имеет "частотное окно" измерения. Допустим, мы имеем идеальный прибор с бесконечным окном

$$\int_{-\infty}^{+\infty} d\omega\, S_{\alpha\beta}(\omega)\, e^{i\omega(T'-T'')} + = S^{(0)}_{\alpha\beta}\, \delta(T'-T'')$$

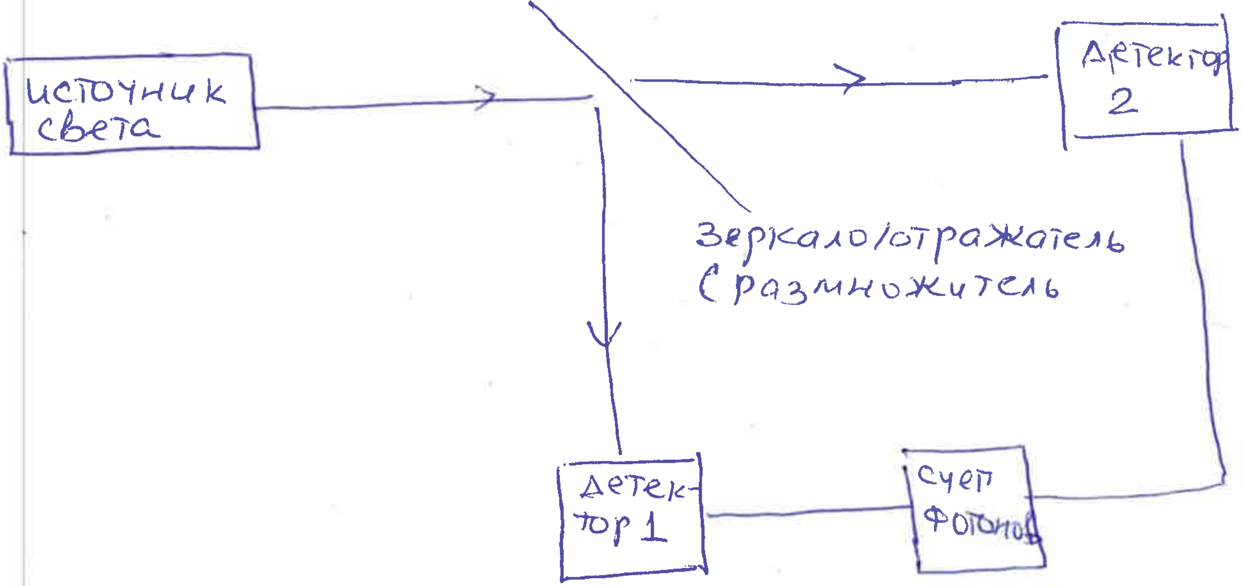
$$\Rightarrow P = \sum_{\alpha\beta}{}' S^{(0)}_{\alpha\beta} \int_{t_0}^{t} G^{(1)}_{\alpha\beta}(T,T)\, dT$$

Результат: прибор меряет значение корреляционной функции, усредненное по времени измерения!

Усложнение задачи: у нас есть два детектора и каждый из них меряет по одному фотону

Что именно измеряется, какая величина?

Пример такого измерения: измерения на совпадения



Фотон в детекторе 2 считается только если есть фотон в детекторе 1 (измерение совпадений

Причем, время измерения в детекторах регулируется D1 - в t, D2 - в t+T.

⑩

$\Rightarrow$ Нам нужно решить задачу взаимодействия с двумя атомами, 1-атом и 2-атом

Решение точно такое же

Гамильтониан системы $\hat{H} = \hat{H}_M + \hat{H}_F + \hat{H}_I$

$\hat{H}_M = \hat{H}_{M_1} + \hat{H}_{M_2}$ $\hat{H}_{M_1}$ - атом 1

$\hat{H}_{M_2}$ - атом 2

$H_I = \hat{H}_{I_1} + \hat{H}_{I_2}$ $\hat{H}_{I_1} = -q\vec{r}_1 \cdot \vec{E}(r_1)$

$\hat{H}_{I_2} = -q r_2 \vec{E}(r_2)$

Решаем точно также по теории возмущений

Различие: теперь нам нужно получить решение не в первом порядке по возмущению, а во втором! Причина: оба атома меняют состояние.

$$|\psi\rangle = e^{-\frac{i\hat{H}_0 t}{\hbar}} |\Phi\rangle \qquad \hat{H}_0 = \hat{H}_{M_1} + \hat{H}_{M_2} + \hat{H}_F$$

$$i\hbar\frac{d}{dt}|\Phi\rangle = \hat{\tilde{H}}_I(t)|\Phi\rangle \qquad \hat{\tilde{H}}_I(t) = e^{\frac{i\hat{H}_0 t}{\hbar}}(\hat{H}_{I_1} + \hat{H}_{I_2}) \times e^{-\frac{iH_0 t}{\hbar}}$$

$$|\Phi\rangle = |\Phi_0\rangle + \lambda|\Phi_1\rangle + \lambda^2|\Phi_2\rangle + \dots$$

Решение:

λ^0: $i\hbar \frac{d}{dt}|\varphi_0\rangle = 0 \Rightarrow |\varphi_0\rangle = |G_1, G_2, i\rangle$

оба атома в основном состоянии

λ^1: $i\hbar \frac{d}{dt}|\varphi_1\rangle = \hat{\tilde{H}}_I(t)|\varphi_0\rangle$

$$\Rightarrow |\varphi_1\rangle = -\frac{i}{\hbar}\int_{t_0}^{t} \hat{\tilde{H}}_I(\tau)\,d\tau\,|\varphi_0\rangle$$

λ^2: $i\hbar \frac{d}{dt}|\varphi_2\rangle = \hat{\tilde{H}}_I(t)|\varphi_1\rangle$

$$\Rightarrow |\varphi_2\rangle = -\frac{i}{\hbar}\int_{t_0}^{t} \hat{\tilde{H}}_I(\tau)|\varphi_1\rangle\,d\tau =$$

$$= -\frac{1}{\hbar^2}\int_{t_0}^{t} d\tau\,\hat{\tilde{H}}_I(\tau)\int_{t_0}^{\tau} d\tau'\,\hat{\tilde{H}}_I(\tau')\,|\varphi_0\rangle =$$

$$= -\frac{1}{\hbar^2}\int_{t_0}^{t} d\tau\int_{t_0}^{t} d\tau'\,\hat{\tilde{H}}_I(\tau)\,\hat{\tilde{H}}_I(\tau')\,|\varphi_0\rangle =$$

трюк $= -\frac{1}{2\hbar^2}\int_{t_0}^{t} d\tau\int_{t_0}^{t} d\tau'\,\hat{T}\left[\hat{\tilde{H}}_I(\tau)\,\hat{\tilde{H}}_I(\tau')\right]|\varphi_0\rangle$

оператор упорядочивания по времени

$$\hat{T}\left[f(\tau)f(\tau')\right] = \begin{cases} f(\tau)f(\tau') & \tau > \tau' \\ f(\tau')f(\tau) & \tau < \tau' \end{cases}$$

Зачем нам нужен 2-ой порядок возмущени

Потому что финальное состояние детекторов возбужденное

$$|f, E_1, E_2\rangle = |f\rangle \otimes |E_1\rangle \otimes |E_2\rangle$$

и тогда амплитуда перехода получается

$$\alpha_{fE_1E_2} = \langle f, E, E_2 | \Phi\rangle$$

Давайте возьмем первый порядок теории возмущений

$$|\Phi_1\rangle = -\frac{i}{\hbar}\int_{t_0}^{t} \hat{\tilde{H}}_I(\tau)\, d\tau\, |G_1, G_2, i\rangle$$

$$\langle f, E_1, E_2 | \Phi_1\rangle = -\frac{i}{\hbar}\int_{t_0}^{t} \langle f, E_1, E_2 | \hat{\tilde{H}}_I(\tau) | i, G_1 G_2\rangle\, d\tau$$

$$\langle f, E_1, E_2 | \hat{\tilde{H}}_{I_1} + \hat{\tilde{H}}_{I_2} | i, G_1, G_2\rangle = \langle f, E_1 E_2 | \hat{\tilde{H}}_{I_1} | i, G_1,$$

$$+ \langle f, E_1, E_2 | \hat{\tilde{H}}_{I_2} | i, G_1, G_2\rangle$$

$$\hat{\tilde{H}}_{I_1} = -q\hat{r}_1 \cdot \hat{E}(r_1) \Rightarrow \{\langle f| \otimes \langle E_1| \otimes \langle E_2|\}\, \hat{H}_I \{|i\rangle \otimes$$

$$|G_1\rangle \otimes |G_2\rangle\} =$$

$$= \{\langle f| \otimes \langle E_1|\}\, \hat{\tilde{H}}_{I_1} \{|G_1\rangle \otimes |i\rangle\}\, \underbrace{\langle E_2 | G_2\rangle}_{=0}$$

Таким образом, все эти матричные элементы равны 0

Амплитуда перехода

$$\alpha_{fE_1E_2} = \langle f E_1 E_2 | \Phi_2 \rangle = -\frac{1}{2\hbar^2}\int\limits_{t_0}^{t} d\tau_1 \int\limits_{t_0}^{t} d\tau_2 \times$$

$$\times \langle f E_1 E_2 | \hat{T}\left[\hat{\tilde{H}}_I(\tau_1)\hat{\tilde{H}}_I(\tau_2)\right] | i G_1 G_2 \rangle$$

$$\hat{\tilde{H}}_I(\tau_1)\hat{\tilde{H}}_I(\tau_2) = \left(\hat{\tilde{H}}_{I_1}(\tau_1) + \hat{\tilde{H}}_{I_2}(\tau_1)\right)\left(\hat{\tilde{H}}_{I_1}(\tau_2) + \hat{\tilde{H}}_{I_2}(\tau_2)\right)$$

$$= \hat{\tilde{H}}_{I_1}(\tau_1)\hat{\tilde{H}}_{I_1}(\tau_2) + \hat{\tilde{H}}_{I_2}(\tau_1)\hat{\tilde{H}}_{I_2}(\tau_2) +$$

$$+ \hat{\tilde{H}}_{I_1}(\tau_1)\hat{\tilde{H}}_{I_2}(\tau_2) + \hat{\tilde{H}}_{I_2}(\tau_1)\hat{\tilde{H}}_{I_1}(\tau_2)$$

$\langle f E_1 E_2 | \hat{\tilde{H}}_{I_1}(\tau_1)\hat{\tilde{H}}_{I_2}(\tau_2) | i G_1 G_2 \rangle = 0$ та же причи

$\langle f E_1 E_2 | \hat{\tilde{H}}_{I_2}(\tau_1)\hat{\tilde{H}}_{I_2}(\tau_2) | i G_1 G_2 \rangle = 0$ что и в первом порядке

остается 2 других члена, они дают одинаковый вклад из-за перестановки + временное упорядочивание. Таким образом

$$\alpha_{fE_1E_2} = \cancel{\langle f E_1 E_2} -\frac{1}{\hbar^2}\int\limits_{t_0}^{t} d\tau_1 \int\limits_{t_0}^{t_1} d\tau_2 \langle f E_1 E_2 | \hat{T}\left[\hat{\tilde{H}}_{I_2}(\tau_1)\hat{\tilde{H}}_{I_2}(\tau_2)\right] | i G_1 G_2$$

/ исчезла "2"

Далее вычисления идут точно также как и ранее. Ответ получается такой для вероятности измерения

$$P = \sum_{\substack{\alpha\\\beta\\\gamma\\\delta}} \int_{t_0}^{t} dT_1' \int_{t_0}^{t_0} dT_1'' \int_{t_0}^{t} dT_2' \int_{t_0}^{t} dT_2'' \times S^{(1)}_{\alpha\beta}(T_1'-T_1'') \times S^{(2)}_{\gamma\delta}(T_2'-T_2''$$

$$\times G^{(2)}_{\alpha\beta\gamma\delta}(r_1; T_1', T_1''; r_2 T_2' T_2'')$$

Здесь $S^{(1)}_{\alpha\beta}(t)$ и $S^{(2)}_{\gamma\delta}(t)$ – введенные ранее функции прибора, а $G^{(2)}$ – корреляционная функция состояния света 2-го порядка

$$G^{(2)}_{\alpha\beta\gamma\delta}(r_1 T_1' T_1'' r_2 T_2' T_2'') = \langle i | \hat{\tilde{E}}^{(-)}_{\beta}(r_1 T_1'') \hat{\tilde{E}}^{(-)}_{\delta}(r_2 T_2'') \times$$

$$\times \hat{\tilde{E}}^{(+)}_{\gamma}(r_2 T_2') \hat{\tilde{E}}^{(+)}_{\alpha}(r_1 T_1') | i \rangle$$

Таким образом, меряется корреляционная функция второго порядка (если производятся измерения на совпадение)

Обычно измеряется только одна мода (количество мод ограничивается оптической конфигурацией)

Выражения для одной моды

$$\hat{\tilde{E}}^{(+)}(r_1,T_1) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0}}\,\vec{e}(r_1)\,\hat{a}(T_1) \qquad [\,\hat{a}(T_1)=\hat{a}e^{-i\omega T_1}$$

$$G^{(2)}_{\alpha\beta\gamma\delta}(r_1 T_1' T_2''; r_2 T_2' T_2'') = A_{\alpha\beta\gamma\delta}(r_1 r_2)\ \times$$

$$\times\ \underbrace{\langle i|\,\hat{a}^+(T_1'')\,\hat{a}^+(T_2'')\,\hat{a}(T_2')\,\hat{a}(T_1')\,|i\rangle}_{\text{Временная корреляция}}$$

Далее, допустим мы имеем идеальный прибор с бесконечным спектром ⟶ он меряет мгновенно

$$S(t'-t'') \to S\,\delta(t'-t'')$$

Допустим, мы уста новим заслонки которые позволяют нам мерять в заданное время: $M_1(t)$ – на детекторе од

$M_2(t)$ – на детекторе 2

$$\Rightarrow\ P = \int_{t_0}^{t} dT_1 \int_{t_0}^{t} dT_2\ \underbrace{S^{(1)}_{\alpha\beta} S^{(2)}_{\delta\gamma} A_{\alpha\beta\delta\gamma}(r_1 r_2)}_{\text{не зависит от времени}} \cdot \underbrace{M_1(T_1)\,M_2(T_2)}_{\text{заслонки}}$$

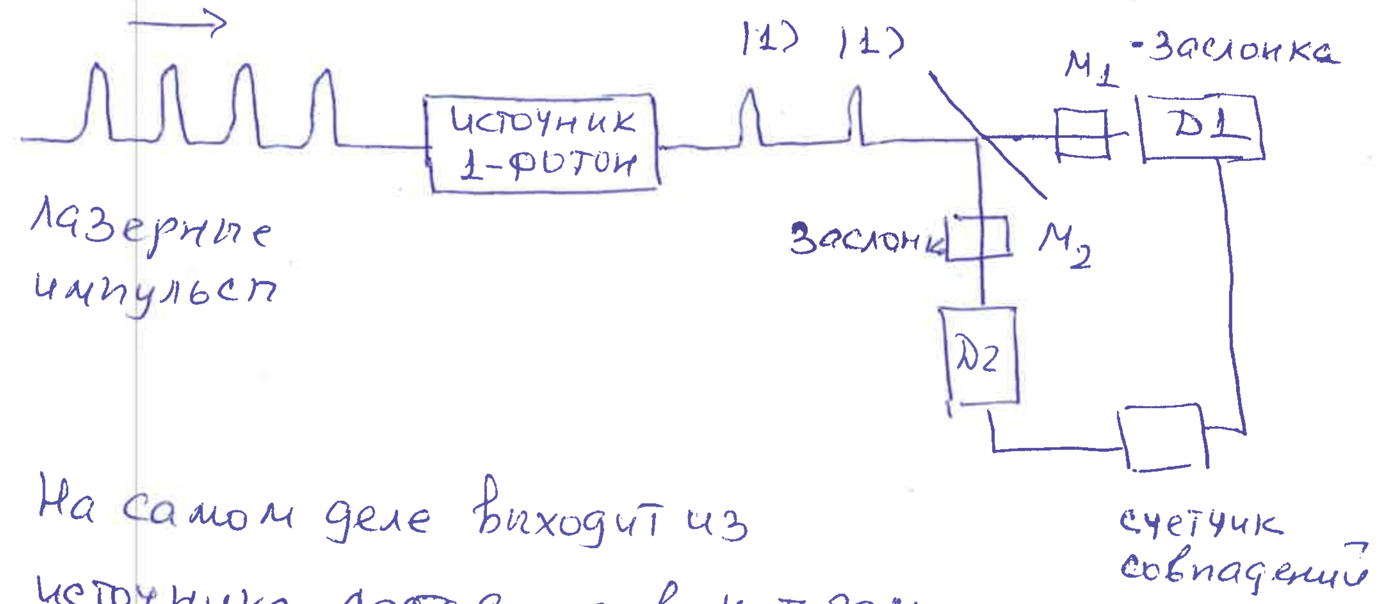
$$\times\ \underbrace{G^{(2)}(T_1 T_2; T_2 T_1)}_{\langle i|\,\hat{a}^+(T_1)\,\hat{a}^+(T_2)\,\hat{a}(T_2)\,\hat{a}(T_1)\,|i\rangle}$$

То есть, измеряя совпадения регистрации фотонов в детекторах 1 и 2, мы на самом деле меряем корреляционную функцию $G^{(2)}(T_1,T_2)$. – двух-фотонную корреляцию.

Зачем она нужна?

Эта фуккция может много рассказать о состоянии света.

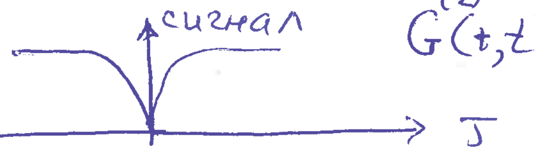
Пример 1: источник одно-фотонных импульсов
Проверим его качество (чистоту однофотонного состояния)



На самом деле выходит из источника состояние, в котором есть любое число фотонов

$$|\psi_{выход}\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle + \dots$$

измеряем совпадения – корреляцию $G^{(2)}(t,t+\tau)$ в зависимости от τ



$G^{(2)}(t,t) = 0$! для хорошего источника

Давайте посчитаем для состояния $|1\rangle$

$$G^{(2)}(t,t) = \underbrace{\langle i|}_{\langle 1|}\, a^+(t)\,\hat{a}^+(t)\,a(t)\,a(t)\underbrace{|i\rangle}_{1} = 0$$

Таким образом если при совпадении времен мы меряем не ноль, то это означает, что состояние содержит много-фотонные вклады. (излучение)

Пример 2: давайте рассмотрим нормированную кореляционную фукцию

$$g^{(2)}(t_1,T_2) = \frac{G^{(2)}(T_1,T_2)}{G^{(1)}(T_1)\,G^{(2)}(T_2)}$$

$$G^{(1)}(T_1) = \langle i|\,\hat{a}^+(T_1)\,\hat{a}(T_1)\,|i_1\rangle$$

Посчитаем значение этой фукции для разных состояний света: при $T_1 = T_2$

а) ~~когда~~ фоковское состояние $|n\rangle$

$$G^{(2)}(J_1=J_2) = \langle n|\,a^+(T)\,a^+(T)\,a(T)\,a(T)\,|n\rangle =$$

$$= \langle n|\,a^+\underbrace{a^+a}_{aa^+-1}\,a\,|n\rangle = \langle n|a^+aa^+a|n\rangle - \langle n|a^+a|n\rangle =$$

$$= n^2 - n = n(n-1)$$

$$G^{(1)}(T) = \langle n|a^+a|n\rangle = n$$

$$g^{(2)}(T,T) = \frac{n^2-n}{n^2} = 1 - \frac{1}{n} < 1$$

b) Когерентное состояние $|\alpha\rangle$

$$G^{(2)}(T,T) = \langle\alpha|a^+a^+aa|\alpha\rangle = |\alpha|^4$$

$$G^{(1)}(T) = \langle\alpha|a^+a|\alpha\rangle = |\alpha|^2$$

$$g^{(2)}(T,T) = \frac{|\alpha|^4}{|\alpha|^4} = 1$$

c) Температурное состояние

В этом случае мы не можем дать чистое квантовое состояние, а только матрицу плотности

$$\hat{\rho}_T = \sum_{n=0}^{+\infty} \rho_n |n\rangle\langle n| \qquad \rho_n = \frac{1}{Z} e^{-\frac{E_n}{T}} = \frac{1}{Z} e^{-\frac{\hbar\omega n}{T}}$$

Тогда $G^{(2)}(T,T) = \sum_{n=0}^{+\infty} \rho_n \langle n| a^+(T) a^+(T) a(T) a(T$

$|n$

$$= \sum_{n=0}^{+\infty} \frac{1}{Z} e^{-\frac{\hbar\omega}{T}n} (n^2-n) = \frac{1+e^{\frac{\hbar\omega}{T}}}{(e^{\frac{\hbar\omega}{T}}-1)^2} - \frac{1}{e^{\frac{\hbar\omega}{T}}-1}$$

$$G^{(1)}(T) = \sum_{n=0}^{+\infty} \frac{1}{Z} e^{-\frac{\hbar\omega}{T}n} \underbrace{\langle n|a^+a|n\rangle}_{n} =$$

$$= \sum_{n=0}^{+\infty} \frac{1}{Z} e^{-\frac{\hbar\omega}{T}n}\, n = \frac{1}{e^{\frac{\hbar\omega}{T}}-1}$$

$$g^{(2)}(t,t) = 2$$

Что будет, когда $\tau_1 - T_2 \to \infty$

$$G^{(2)}(t, t+T) \xrightarrow[T\to\infty]{} \ ?$$

Ожидается, что корреляции между фотонами исчезнут, и выражение для коррелятора факторизуется

$$G^{(2)}(t,t+T) = \langle \hat{a}^+(t)\,\hat{a}^+(t+T)\,\hat{a}(t+T)\,\hat{a}(t)\rangle$$

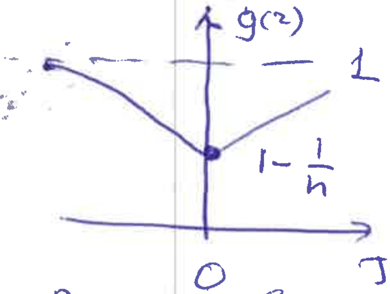
$$\simeq \langle \hat{a}^+(t)\,\hat{a}(t)\rangle \langle \hat{a}^+(t+T)\,\hat{a}(t+T)\rangle$$

$$= G^{(1)}(t)\,G^{(1)}(t+T)$$

$$\Rightarrow \quad g^{(2)}(t,T) = \frac{G^{(2)}(t,T)}{G^{(1)}(t)\,G^{(1)}(T)} = 1$$

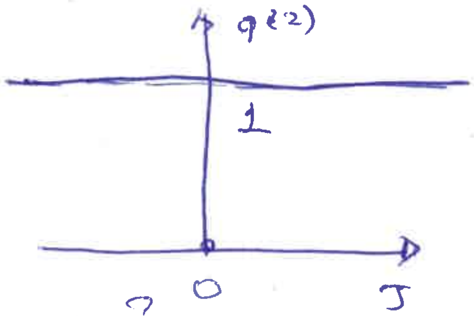
Таким образом, ожидается следующее поведение $g^{(2)}(t,t+T)$

Фоковское состояние



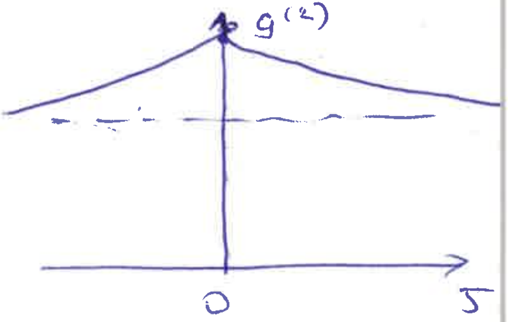
расталкивание
anti-bunching

Когерентное состояние



нейтральное

Температурное состояние



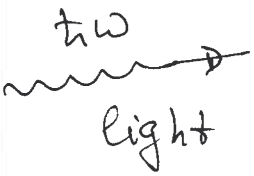
концентрация
bunching

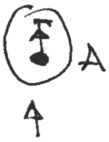
Wednesday 621 16.-

Measurements Friday FAN (101) 12^{15} ①

See other side

What and how we measure in optics





Atom changes state

measuring device (atom interacting with light)

The Hamiltonian

$$\hat H = \hat H_M + \hat H_F + H_I$$

$\hat H_M$ (atom, Coulomb field, electron)

$\hat H_F = \hbar\omega a^+ a$

Heisenberg picture

$\hat H_I = -e\hat{r}\cdot\hat{\underline{E}}(r,t)$

$$H_I^{*}(t) = e^{i(H_M+H_F)t/\hbar}\,\hat H_I\, e^{i(H_M+H_F)t/\hbar} = -e\hat{r}(t)\,\hat{\underline{E}}(r,t)$$

Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H_I(t)|\psi\rangle$$ (interaction representation)

Solution (formal)

$$|\psi_t\rangle = \hat U(t,t_0)|\psi_{t_0}\rangle$$ $\hat U(t,t_0)$ - evolution operator

$$\hat U(t,t_0) = \left(\hat{\mathbb{1}} - \frac{i}{\hbar}\int_{t_0}^{t}\hat H_I(t')\,dt'\right)$$ perturbation theory

$$\left\{ \begin{array}{l} H_I \to \lambda H_I, \quad |\psi\rangle = |\psi_0\rangle + \lambda|\psi_1\rangle + \lambda^2|\psi_2\rangle \\ U = \hat{\mathbb{1}} + \lambda U_1 + \lambda^2 U_2 \end{array} \right\}$$

Atom initially is in the groung state, ligth state$|i\rangle$: $|G,i\rangle$

finally in excited state, light state $|f\rangle$ $|A,f\rangle$

Transition $\overset{\text{matrix}}{\vee}$ elements

$|i\rangle$ - intial field state

$|f\rangle$ - final

$$\langle Af|\hat{U}(t,t_0)|Gi\rangle = -\frac{i}{\hbar}\int_{t_0}^{t}\langle Af|\overset{\hat{H}_I(t')}{\cancel{\hat{U}(t',t_0)}}|Gi\rangle dt' =$$

$$= +\frac{i}{\hbar}\int_{t_0}^{t} dt' \langle A|e\hat{\vec{r}}(t')|G\rangle\langle f|\vec{E}(r,t')|i\rangle$$

$$\langle A|\overset{e}{\vec{r}}(t)|G\rangle = \langle A|e^{+iH_A t'/\hbar}\,\vec{r}(0)\,e^{-i\frac{H t'}{\hbar}}|G\rangle = e^{i\omega_{AG}t}\vec{M}_{AG}$$

$\omega_{AG} = E_A - E_G/\hbar$

M_{AG} - matrix eleme

$$\Rightarrow \langle Af|\hat{U}(t,t_0)|Gi\rangle = \frac{i\vec{M}_{AG}}{\hbar}\int_{t_0}^{t} e^{i\omega_{AG}t'}\langle f|\hat{\vec{E}}(r,t')|i\rangle dt'$$

$$\hat{\vec{E}} = \vec{E}^{(+)} \cancel{(} + \vec{E}^{(-)} \qquad \hat{\vec{E}}^{(+)} = \hat{\vec{E}}^{(-)*}$$

$$\hat{\vec{E}}^{(+)} = \sum_n \vec{e}_n\cdot\hat{a}_n e^{-i\omega_n t} \qquad \hat{\vec{E}}^{(-)} = \sum_n \vec{e}_n^{\,*} a_n^{+} e^{+i\omega_n t}$$

Consider.

$$\int_{t_0}^{t} e^{i(\omega_{AG}-\omega_n)t'}dt'$$ most contributing @ $\omega_{AG}\simeq\omega_n$.

$$\left\{\begin{array}{l}\text{If } t_0\to-\infty\\ \quad\;\; t\to+\infty\end{array}\right\} \Rightarrow \int_{-\infty}^{+\infty}dt' \to 2\pi\delta(\omega_{AG}$$

Terms with $\hat{E}^{(-)}(t)$ are negligible because

$$\int_{t_0}^{t} e^{i(\omega_{AG}+\omega_n)t'}dt'$$ - fast oscillating function $\to 0$.

This is equivalent to one photon absorbed (not emitted)

RWA!

$\Rightarrow \quad \langle f|\hat{E}|i\rangle \Rightarrow \langle f|\hat{E}^{(+)}|i\rangle$ ③

We want to calculat probability

$$W = \sum_f |\langle f A|\hat{u}|i G\rangle|^2 =$$

(sum over al possible final states of ligth)

$$= \frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \, e^{i\omega_{AG}(t'-t'')} \, M^{\nu}_{AG} M^{\eta *}_{AG} \times$$

$$\times \langle i|\hat{E}^{(-)}_{\eta}(t')\underbrace{\sum |f\rangle\langle f|}_{1}\hat{E}^{(+)}_{\nu}(t'')|i\rangle =$$

$$\left[\sum_f |f\rangle\langle f| = \hat{1}\right]$$

$$= \frac{1}{\hbar^2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \, e^{i\omega_{AG}(t'-t'')} \, M^{\nu}_{AG} M^{\eta *}_{AG} \cdot \underbrace{\langle i|\hat{E}^{(-)}_{\eta}(r,t')\hat{E}^{(+)}_{\nu}(r,t'')|i\rangle}_{G^{(1)}_{\eta\nu}(r,t',r,t'')}$$

$$\langle i|\hat{A}|i\rangle \to Tr\{\hat{\rho}\hat{A}\} = \langle\hat{A}\rangle$$

$G^{(1)}(rt', rt'')$ - correlation function for the fields.

One can write this as

$$W = \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \underbrace{K^{AG}_{\eta\nu}(t'-t'')}_{\text{measuring function of apparatus}} \overbrace{G^{(1)}_{\eta\nu}(rt', rt'')}^{\text{correlation function of the field}}$$

$W(\overset{G \to A}{G})$ the probability of atom to change the state from $G \to A$

If we do not care abote this ~~probability~~ state

$$P = \sum_A R^A W(G \to A)$$ R^A - is the weigth function

$$S_{\nu\eta}(\omega) = \frac{2\pi}{\hbar^2}\sum_A R_A M^{\nu}_{AG} M^{\eta *}_{AG}\,\delta(\omega - \omega_{AG})$$

$$P(t) = \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \int d\omega\, S_{\nu\eta}(\omega)\, e^{i\omega(t'-t'')}\, G^{(1)}_{\eta\nu}(rt', rt'')$$

Case of wide frequency detector $S_{\eta\nu}(\omega) = const$

$$\int_{-\infty}^{+\infty} d\omega\, S_{\eta\nu}(\omega)\, e^{i\omega(t'-t'')} = S_{\eta\nu}\cdot\delta(t'-t'')$$

$$\Rightarrow P(t) = S_{\eta\nu}\int_{t_0}^{t} G^{(1)}_{\eta\nu}(rt', rt')\,dt'$$

Then the rate

$$\boxed{[W =]\ \frac{dP}{dt} = S_{\eta\nu}\, G^{(1)}_{\eta\nu}(r t, r t)}$$

Many detectors

The same model but many atoms

$$\hat{H}_I = -\sum_j \underbrace{e(\hat{\vec{r}}_j(t)\vec{E}(\vec{r}_j, t))}_{H_{jI}(t)}.$$

j - sum over atoms with positions $\vec{r}_j$

Then we need

$$U^{(n)}(t, t_0) = \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \dots \int_{t_0}^{t} dt_n \times T\Big[H_I(t_1)\cdot H_I(t_2)$$

$$\cdot H_I(t_3)\dots\dots H_I(t_n)\Big]$$

$$T[f(t_1)f(t_2)] = \begin{cases} f(t_1)f(t_2)\ t_1 \\ f(t_2)f(t_1)\ t_1< \end{cases}$$

Time ordering operator

If one photon per detector is absorbed

Then

⑤

$$U^{(n)} = \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\left[\prod_{j=1}^{n}\int_{t_0}^{t} dt_j\right] \times T\Big[H_{I,1}(t_1)\cdot H_{I,2}(t_2)\cdots\cdots H_{I,n}(t_n) +$$

$$\underbrace{+ H_{I,2}(t_1)\, H_{I,1}(t_2)\cdots\cdots H_{I,n}(t_n) + \text{All permutations (of } j),\ n!\ \Big]}_{}$$

All contributions are the same because of the time ordering

$$\Rightarrow U^{(n)} = \left(\frac{-i}{\hbar}\right)^n \left[\prod_{j=1}^{n}\left(\int_{t_0}^{t} dt_j\right)\right] T\left[H_{I,1}(t_1)\cdots\cdots H_{I,n}(t_n)\right]$$

$$\approx \left(\frac{-i}{\hbar}\right)^n \left[\prod_{j=1}^{n}\left(\int_{t_0}^{t} dt_j\right)\right] \times e^{i\sum_p \omega^{p}_{AG} t_j} \langle f | E^{(+)}_{\nu_n}(r_n t_n)\, E^{(+)}_{\nu_{n-1}}(r_{n-1} t_{n-1})$$

$$\cdots E^{+}_{\nu_1}(r_1 t_1) | i\rangle \times \left[\prod_{j=1}^{n} M^{\nu_j}_{AG_j}\right]$$

Probability

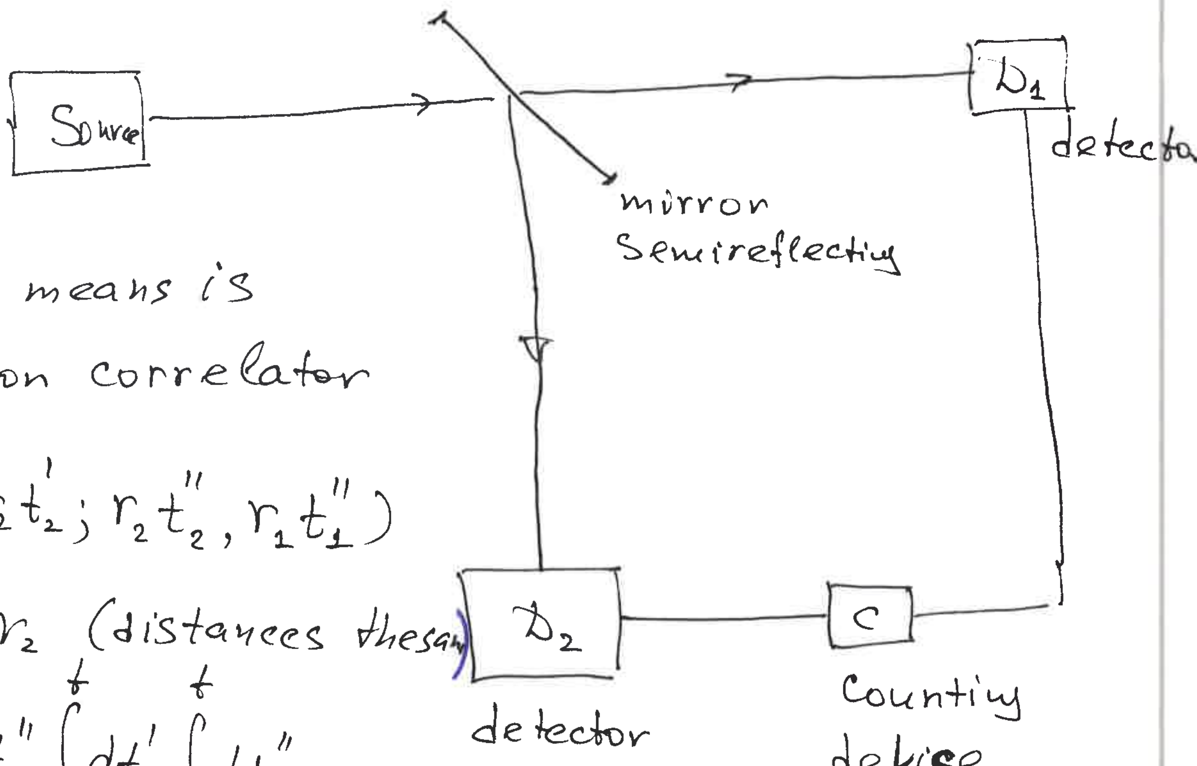
$$p_n(t) = \left[\prod_{j=1}^{n}\int_{t_0}^{t} dt'_j \int_{t_0}^{t} dt''_j\right]\left[\prod_{j=1}^{n} S(t'_j - t''_j)\right] G^{(n)}_{\nu_1\nu_2\ldots\,\eta_1\ldots\eta_n}(r_1 t'_1 \cdots r_n t'_n;\ r_n t''_n \ldots r_1 t''_1)$$

$$G^{(n)}_{\nu_1\nu_2\cdots\nu_n\,\eta_1\cdots\eta_n}(r_1 t'_1 \cdots\cdots r_1 t''_1) = \langle \hat{E}^{(-)}_{\nu_1}(r_1 t'_1)\cdots \hat{E}^{(-)}_{\nu_n}(r_n t'_n)\cdot$$

$$\times \hat{E}^{(+)}_{\eta_n}(r_n t''_n)\cdots E^{(+)}_{\eta_1}(r_1 t''_1)\rangle$$

$$\langle \hat{A}\rangle = \mathrm{Tr}\{\hat{\rho}\cdot\hat{A}\}$$

Example: two detectors (coincidence measurements)



↓

Photons in detector No. 2 (at t_2) are counted only when a photon is counted in detector No 1 (at t_1)

=) what it means is a two photon correlator

$$G^{(2)}_{\substack{\nu_1\nu_2\\ \eta_1\eta_2}}(r_1 t_1', r_2 t_2'; r_2 t_2'', r_1 t_1'')$$

let $r_1 = r_2$ (distances the same)

$$P = \int_{t_0}^{t} dt_1' \int_{t_0}^{t} dt_1'' \int_{t_0}^{t} dt_2' \int_{t_0}^{t} dt_2'' \times$$

$$\times S_{\nu_1\eta_1}(t_1'-t_1'')\, S_{\nu_2\eta_2}(t_2'-t_2'') \cdot G^{(2)}_{\substack{\nu_1\nu_2\\ \eta_1\eta_2}}(\dots)$$

Let $S_{\nu_1\eta_1}(t_1'-t_1'') \simeq S_1\,\delta(t_1'-t_1'')$ (wide frequency detector)

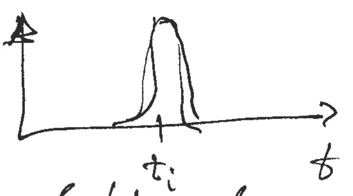
↓

omit indices for clarity

$$P(t) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\; S_1 S_2 \cdot G^{(2)}(t_1, t_2; t_2, t_1) \cdot \underbrace{\mu_1(t_1)}_{\text{"Shutter" for } D_1}\; \underbrace{\mu_2(t_2)}_{\text{"shatter" for } D_2}$$

$\mu_1(t_1)$ opens only at $t_1 = t$

$\mu_2(t_2)$ opens only at $t_2 = t + T$



$$P = S_1 S_2\, G^{(2)}(t, t+T; t+T, t) =$$

$$S_1 S_2\, G^{(2)}(t, t+T)$$ — two time correlation function

$$G^{(2)}(t,t+T) = \langle \underbrace{\hat{E}^{(-)}(t)\, E^{(-)}(t+T)\, \hat{E}^{(+)}(t+T)\, \hat{E}^{(+)}(t}$$

$$\vec{E}^{(+)}_{(r,t)} = \sum_{n\,(\text{modes})} \vec{e}_n(r) \cdot a_n e^{-i\omega_n t}$$

Let only 1 mode be important.

$$\Rightarrow \hat{E}^{(+)}_{(r,t)} \simeq \vec{e}(r) \cdot \hat{a}\, e^{-i\omega t} =: \vec{e}(r) \cdot \hat{a}(t) =$$

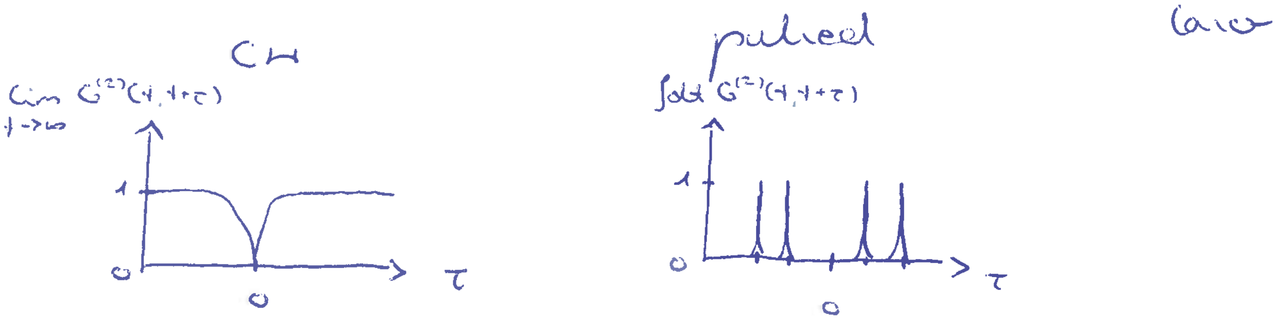
$$\Rightarrow G^{(2)}(t,t+T) \sim \langle \hat{a}^{+}(t)\, \hat{a}^{+}(t+T)\, \hat{a}(t+T)\, \hat{a}(t) \rangle =$$

$$= \mathrm{Tr}\{ \hat{\rho}\, \hat{a}^{+}(t)\, \hat{a}^{+}(t+T)\, \hat{a}(t+T)\, \hat{a}(t) \}$$

Coincidence measurements with two detectors measures two time correlation function.

Hanbury Brown - Triss Setup (HBT)

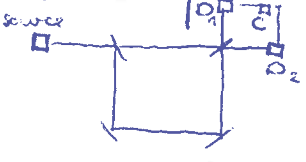
Applications in current research:
Trigger a photon source (e.g. semiconductor QD) with



Zero at / around $\tau = 0$ => perfect single-photon source

Outlook:

Indistinguishability of successively emitted photons via Hong-Ou-Mandel Setup (HOM)



↳ specialized model for the $G^{(2)}(t,t+\tau)$ function

e.g.
Fischer et al.,
New J. Phys. 1
113053 (201

Correlation functions

How the value of the field alt one point (time) is relatad to the field at the outher point (time)

$$\hat{I} = |\hat{E}|^2$$

$$I = \sum_i \langle i | E^-(r,t)\, E^+(r,t) | i \rangle w_i =$$

$$= \mathrm{Tr}\{\hat{\rho}\, E^-(r,t)\, E^+(r,t)\}$$ $\hat{\rho}$ - statistical operator

$$\hat{E}^{(+)} = \sum_i \sqrt{\frac{\hbar\omega_i}{2\varepsilon_0}}\, \vec{e}_i(r,t)\, \hat{a}_i e^{-i\omega_i t}$$

$$\hat{E}^{(-)} = \left(\hat{E}^{(+)}\right)^+$$

Correlation function (first order)

$$G^{(1)}(x,x') = \mathrm{Tr}\{\hat{\rho}\, \hat{E}^{(-)}(x)\, \hat{E}^{(+)}(x')\}$$

$x = (r,t)$

Correlation function of the ~~next~~ higher orders -th order

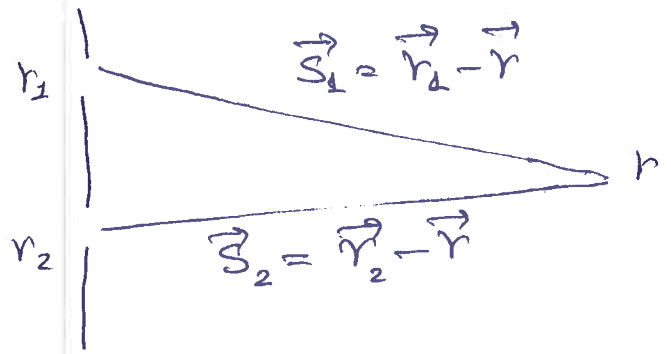
$$G^{(n)}(x_1, x_2 \dots x_n, x_1' x_2' \dots x_n') = \mathrm{Tr}\{\hat{\rho}\, \hat{E}^{(-)}(x_1)\, \hat{E}^{(-)}(x_2) \dots \hat{E}^{(+)}(x_1')\, E^{(+)}(x_2') \dots E^{(+)}(x_n')\}$$

Properties of the n=1 correlation function

$$G^{(1)}(x,x) = \mathrm{Tr}\{\hat{\rho}\, \hat{E}^{(-)}(x)\, \hat{E}^{(+)}(x)\} \geq 0.$$

$$G^{(1)}(x_1,x_1)\,G^{(1)}(x_2,x_2) \geq |G^{(1)}(x_1,x_2)|^2$$

Schwarz inequality



$$E^{(+)}(r,t) = E_1^{(+)}(r,t) + E_2^{(+)}(r,t)$$

$$E_i^{(+)}(r,t) = E_i^{(+)}\left(\vec{r}_i, t - \frac{|\vec{r}_i - \vec{r}|}{c}\right) \frac{e^{i(k-\omega/c)|\vec{r}-\vec{r}_i|}}{|\vec{r}-\vec{r}_i|}$$

$$I(r) = \measuredangle \, \mathrm{Tr}\{\hat{p}\, E^{(-)}(r,t)\, E^{(+)}(r,t)\} =$$

$=$ [illegible]

$$= G^{(1)}(x_1,x_1) + G^{(1)}(x_2,x_2) + 2|G^{(1)}(x_1,x_2)|\cos(\psi)$$

$$G^{(1)}(x_i,x_i) = \cancel{E} \langle \hat{E}^{(+)}(x_i)\, \hat{E}^{(-)}(x_i) \rangle$$

$$G^{(1)}(x_1,x_2) = |G^{(1)}(x_1,x_2)|\, e^{i\psi}$$

Normalized correlation function

$$g^{(1)}(x_1,x_2) = \frac{G^{(1)}(x_1,x_2)}{\sqrt{|G^{(1)}(x_1,x_1)|\,|G^{(1)}(x_2,x_2)|}}$$

Best contrast in experiments

$|g^{(1)}(x_1,x_c)| = 1$ if $|g^{(1)}(x_1,x_2)| \to 0$

no contrast.

Second order correlation function

Photon bunching/anti-bunching

$G^{(1)}$ – amplitude correlations

$G^{(2)}$ – intensity correlations

$$G^{(2)}(x_1,x_2) = \langle \hat{E}^{(-)}(x_1)\hat{E}^{(-)}(x_2)\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)\rangle$$

$$= \langle :\hat{I}(x_1)\hat{I}(x_2):\rangle \qquad : \; : \to \text{normal ordering}$$

Normalized correlator

$$g^{(2)}(x_1,x_2) = \frac{G^{(2)}(x_1,x_2)}{G^{(1)}(x_1,x_1)\,G^{(1)}(x_2,x_2)}$$

Special case of temporal correlations

$x_1 = x_2 = x$ (omit the dependence)

$t_1 = t \quad t_2 = t+T$

$$g^{(2)}(t,T) = \frac{\langle :\hat{I}(t+T)\,\hat{I}(t):\rangle}{\langle :I(t):\rangle\langle :I(t+T):\rangle}$$

Scwarz-inequality for classical fields

$$|\langle :\hat{I}(t)\hat{I}(t+T):\rangle|^2 \leq \langle|\hat{I}(t)|^2\rangle\langle|\hat{I}(t+T)|^2\rangle$$

classical fields

$$\underline{g^{(2)}(T) \leq g^{(2)}(0)}$$

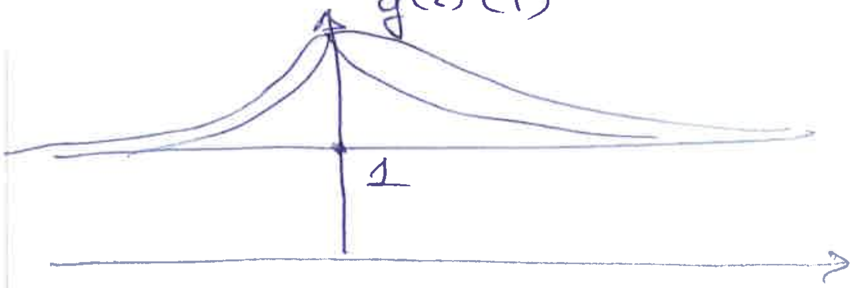
Classical distribution of $|E| = \varepsilon$

$$g^{(2)}(0) = 1 + \frac{1}{\langle \varepsilon^2 \rangle^2} \int P(\varepsilon)\left(\varepsilon^2 - \langle \varepsilon^2 \rangle\right)^2 d\varepsilon$$

$P(\varepsilon)$ – positive probability distribution

$$\Rightarrow \quad \boxed{g^{(2)}(0) \geq 1}$$

Types of distribution



~~ga~~ $g^{(2)}(\tau) = 1 + e^{-\gamma\tau}$ Lorenzian

$g^{(2)}(\tau) = 1 + e^{-\gamma^2\tau^2}$ Gaussian

Coherent field

$$G^{(2)}(\tau) = \varepsilon^{(-)}(t)\, \varepsilon^{(-)}(t+\tau)\, \varepsilon^{(+)}(t+\tau)\, \varepsilon^{(+)}(t)$$

$$\Rightarrow \quad \boxed{g^{(2)}(\tau) \Rightarrow = 1}$$

Quantum fields

Single mode

$$E^{(+)} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0}}\,\vec{e}(r)\,e^{-i\omega t}\,\hat{a}$$

$$g^{(2)}(0) = \frac{\langle a^+ a^+ a a\rangle}{\langle a^+ a\rangle^2} = 1 + \frac{\overset{(\overline{\Delta n})^2}{\cancel{\text{[illegible]}}} - \bar{n}}{\bar{n}^2}$$

$$\overline{\Delta n^2} = \langle \hat{n}^2\rangle - \langle n\rangle^2 = \langle (a^+ a)^2\rangle - \langle a^+ a\rangle^2$$

1) coherent field

$$g^{(2)}(0) = 1$$

2) Fock state

$$g^{(2)} = 1 - \frac{1}{n} < 1$$

3) Temperature distribution

$$\bar{n} = \cancel{kt}\; n_T \qquad \boxed{g^{(2)}(0) = 2}$$

$$\langle (a^+ a)^2\rangle = \frac{1}{Z}\sum_n e^{-\frac{\hbar\omega n}{T}} \cdot n^2 = \cancel{\frac{e^{x}(1+e^{x})}{(e^{x}-1)^3}} =$$

$$= \frac{1+e^{\hbar\omega/T}}{(e^{\hbar\omega/T}-1)^2} \qquad \frac{\langle(\Delta n)^2\rangle - \bar{n}}{\bar{n}^2} =$$

$$\langle \Delta n^2\rangle = \frac{e^{\hbar\omega/T}}{(e^{\hbar\omega/T}-1)^2}$$

~~g~~

$g^{(2)}(0) \geqslant 1$ bunching

$g^{(2)}(0) < 0$ antibunching

---

measuring intensity correlations

one can see if the quantum state

is really quantum or aclassical

Intuitive example

Fock state $|n\rangle$

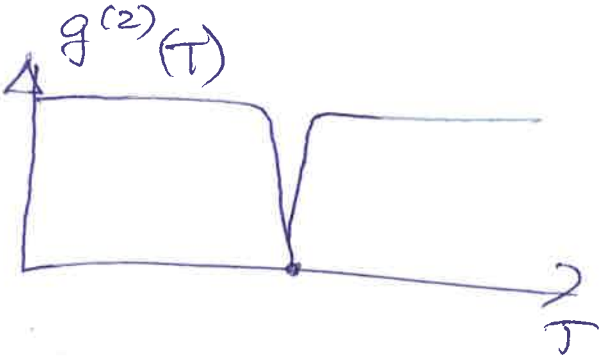
measuring correlation of two intensities

one removes one photon.

~~B~~ $g^{(2)} \sim \frac{n-1}{n} \simeq 1 - \frac{1}{n} < 1.$

If $n=1$ then the state after measurement is $n=0$.

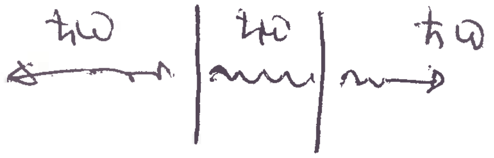
$$\boxed{g^{(2)} = 0.}$$



Потери в квантовой механике

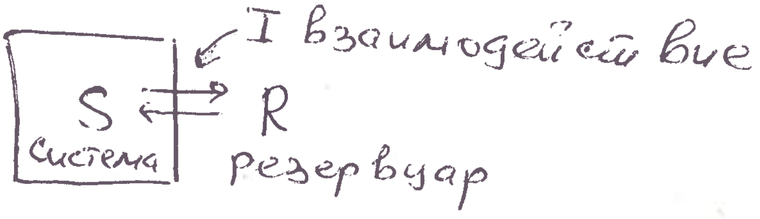
Квантовомеханические свойства определяются уравнением Шрёдингера, которое сохраняет энергию. Откуда же берутся потери? Как их описать в рамках уравнения Шредингера?

Простой пример: неидеальные зеркала в резонаторе → свет "вылетает" из резонатора



Это элементарный пример "открытой" квантовой системы.

Гамильтониан всей системы: $\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_I$



Пример: 1 мода в резонаторе, много мод в резервуаре

$$\hat{H}_S = \hbar\omega a^+ a \qquad H_R = \sum_k \hbar\omega_k b_k^+ b_k$$

$$H_I = \sum_k \hbar\left[g_k^* \hat{b}_k^+ \hat{a} + g_k \hat{a}^+ \hat{b}_k\right]$$

Энергия сохраняется в общей системе S+R, но нас интересует энергия (состояние) системы S (резонатор)

Как можно описать динамику системы

1) Решить полную задачу (трудно)
2) Решить часть задачи "основное уравнение" (требует аппроксимаций)
3) решить модельную задачу (нужно выбрать модель, которая описывает "нужную часть" задачи)

·Решим полную задачу и посмотрим, откуда в принципе возникают потери в квантовой механике.

Решим такую задачу: пусть изначально в резонаторе находится 1 фотон, как это состояние меняется со временем.

a) Общее число фотонов сохраняется

$$\hat N = \hat a^+ \hat a + \sum_k \hat b_k^+ \hat b_k \qquad [\hat H, \hat N] = 0$$

$$\hat H = \hbar\omega \hat a^+ \hat a + \sum_k \hbar\omega_k \hat b_k^+ \hat b_k + \sum \hbar\left[g_k \hat a^+ \hat b_k + g_k^* \hat b_k^+ \hat a^*\right]$$

энергия тоже сохраняется

$$\hat H|\psi\rangle = i\hbar\frac{d}{dt}|\psi\rangle \qquad |\psi\rangle = e^{-\frac{iEt}{\hbar}}|\varphi\rangle \Rightarrow \hat H|\varphi\rangle = E|\varphi$$

если мы выбрали состояние с энергией E, то это же состояние и останется. Более того, средняя энерг

$$E = \langle\psi|\hat H|\psi\rangle \quad \frac{d}{dt}E = \frac{d}{dt}\langle\psi|\hat H|\psi\rangle = \left[\frac{d}{dt}\langle\psi|\right]\hat H|\psi\rangle + \cancel{d}\langle\psi|\hat H\left[\frac{d}{dt}|\psi\rangle\right] = 0$$

Что же такое "потери"? Посмотрим на состояния возможные в этой системе. Если первоначально был один фотон, то один фотон и остается.

Состояния с одним фотоном обозначение

$$|\psi_1\rangle \Rightarrow |1_s, 0_{k_1} 0_{k_2} \ldots\ldots 0_{k_N} \ldots\ldots\rangle \equiv |\Phi\rangle$$

$$|0_s, 1_{k_1}, 0_{k_2}, 0_{k_3} \ldots 0_{k_N} \ldots\rangle \equiv |\Phi_{k_1}\rangle$$

$$|0_s, 0_{k_1}, 1_{k_2}, 0_{k_3} \ldots 0_{k_N} \ldots\rangle = |\Phi_{k_2}\rangle$$

$$\vdots \qquad \vdots$$

Состояние системы — сумма всех состояний

$$|\psi\rangle = \alpha|\Phi\rangle + \sum_k \beta_k |\Phi_k\rangle.$$

$$\langle\Phi|\Phi_k\rangle = 0$$
$$\langle\Phi_k|\Phi_{k'}\rangle = \delta_{kk'}$$
$$\langle\Phi|\Phi\rangle = 1$$

Решение уравнения Шредингера

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$$

$$i\hbar\frac{d}{dt}|\psi\rangle = i\hbar\left[\dot{\alpha}|\Phi\rangle + \sum_k \dot{\beta}_k|\Phi_k\rangle\right] = \hat{H}\left[\alpha|\Phi\rangle + \sum_k \beta_k|\Phi_k\rangle\right]$$

Проецируем уравнение на базис $|\Phi\rangle, |\Phi_k\rangle$

$$\langle\Phi|i\hbar\frac{d}{dt}|\psi\rangle = i\hbar\dot{\alpha} \,; \quad \langle\Phi|\hat{H}_S|\psi\rangle = \langle\Phi|\omega\hbar\hat{a}^+\hat{a}|\psi\rangle =$$

$$= \hbar\omega\alpha \,; \quad \langle\Phi|\hat{H}_R|\psi\rangle = \langle\Phi|\sum_k \hbar\omega_k\hat{b}_k^+\hat{b}_k|\psi\rangle = 0$$

$$\langle\Phi|\hat{H}_I|\psi\rangle = \langle\Phi|\sum_k \hbar[g_k\hat{a}^+\hat{b}_k + g_k^*\hat{b}_k^+\hat{a}]|\psi\rangle =$$

$$= \sum_k \hbar g_k\beta_k \,;$$

$\langle \Phi_k | \hat{H}_S | \psi \rangle = 0; \quad \langle \Phi_k | \hat{H}_R | \psi \rangle = \hbar \omega_k \beta_k$

$\langle \Phi_k | \hat{H}_I | \psi \rangle = \hbar g_k^* \alpha$

$\hat{a}^+ \hat{a} | \Phi \rangle = | \Phi \rangle \quad \hat{b}_k^+ \hat{b}_k | \Phi_{k'} \rangle = \delta_{kk'} | \Phi_{k'} \rangle$

$\hat{a}^+ \hat{b}_k | \Phi_{k'} \rangle = | \Phi \rangle \delta_{kk'}$

Уравнения:

$$\begin{cases} i\dot{\alpha} = \omega \alpha + \sum_k g_k \beta_k \\ i\dot{\beta}_k = \omega_k \beta_k + g_k^* \alpha \end{cases}$$

Решение уравнения 2 методом переменных коэффициентов

$$\beta_k = e^{-i\omega_k t} \tilde{\beta}_k$$

$$i e^{-i\omega_k t} \dot{\tilde{\beta}}_k + e^{-i\omega_k t} \cancel{\omega_k \tilde{\beta}_k} = \cancel{\omega_k \beta_k} e^{-i\omega_k t} + g_k^* \alpha$$

$$\Rightarrow \dot{\tilde{\beta}}_k = -i g_k^* e^{i\omega_k t} \alpha \Rightarrow \tilde{\beta}_k = \tilde{\beta}_k(t_0) - i g_k^* \int_0^t e^{i\omega_k \tau} \alpha(\tau) d$$

$\tilde{\beta}_k(0)$ = начальное условие при $t = t_0 = 0$

$$\Rightarrow \beta_k(t) = e^{-i\omega_k t} \tilde{\beta}_k(0) - i g_k^* \int_0^t e^{-i\omega_k (t-\tau)} \alpha(\tau) d\tau$$

Подставим это решение в уравнение 1

$$\dot{\alpha} = -i\omega\alpha - i e^{-i\omega_k t} \tilde{\beta}_k(0) - \int_0^t \sum_k |g_k|^2 e^{-i\omega_k (t-\tau)} \alpha(\tau) d$$

Граничное условие

$$|\psi\rangle_{t=0} = \alpha(0) |\Phi\rangle + \sum_k \beta_k(0) |\Phi_k\rangle = |\Phi\rangle$$

(один фотон в резонаторе)

$$\Rightarrow \alpha(0) = 1 \quad \beta_k(0) = 0$$

Получается уравнение для α

$$\dot{\alpha} = -i\omega\alpha - \int_0^t K(t-\tau)\,\alpha(\tau)\,d\tau$$

где $K(t) = \sum_k |g_k|^2 e^{-i\omega_k t}$

Это интегро-дифференциальное уравнение. Его можно решить методом Лапласа, но интереснее просто посмотреть какие будут решения.

Замечание:

производная $\dot{\alpha}$ зависит от решения уравнения $\alpha(\tau)$ в прошлые моменты времени $\Rightarrow$ система с памятью.

Приближение 1: память очень короткая

Другими словами изменение $\alpha(t)$ со временем происходит очень медленно

Тогда:

$$\int_0^t K(t-\tau)\alpha(\tau)d\tau \simeq \alpha(t)\int_0^t K(t-\tau)\,d\tau$$

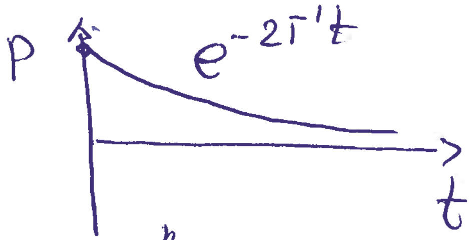
$$= \alpha(t)\int_0^t K(\tau)d\tau \simeq \alpha(t)\underbrace{\int_0^{+\infty} K(\tau)d\tau}_{\Gamma = \Gamma' + i\Gamma''}$$

$$\dot{\alpha} = -i\omega\alpha - \Gamma\alpha \equiv -\Omega\alpha \qquad \Omega = i\omega + \Gamma$$

$$\Rightarrow \alpha(t) = e^{-\Omega t}\underbrace{\alpha(t{=}0)}_{=1} = e^{-i(\omega+\Gamma'')t}\,e^{-\Gamma' t}$$

Вероятность, что в резонаторе 1 photon

$$P_1 = |\alpha|^2 = e^{-2\Gamma' t}$$



Если "памяти нет" (короткая память), то система теряет фотоны экспоненциально

$$\Gamma' = \mathrm{Re}\int_0^{+\infty} K(\tau)\,d\tau = \mathrm{Re}\int_0^{\infty} \sum_k |g_k|^2 e^{+i(\omega - \omega_k)\tau} d\tau$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \sum_k{}' |g_k|^2 \underbrace{e^{+i(\omega-\omega_k)\tau}}_{2\pi\,\delta(\omega-\omega_k)} d\tau = \frac{2\pi}{2}\sum_k |g_k|^2 \delta(\omega-\omega_k$$

$$\underbrace{2\Gamma = 2\pi \sum_k{}' |g_k|^2 \delta(\omega-\omega_k)}_{\text{Золотое правило Ферми!}}$$ – скорость потери фотонов

Итак: Потери в системе связаны с памятью

Короткая память ≡ быстрая релаксация
=> потери энергии (числа фотонов)

Бесконечно короткая память (нет памяти)
=> потери даются правилом Ферми!

Замечание: мы выбрали константу связи $\hbar g_k$
Если константа связи просто g_k, то выражение будет $\frac{|g_k|^2}{\hbar^2}$ как в оригинальной формуле Ферми.

Приближение 2 : бесконечная память

$$K(t) = G \text{ (константа)}$$

$$\Rightarrow \dot{\alpha} = -i\omega\alpha - G\int_0^t \alpha(\tau)\,d\tau$$

$$\Rightarrow \ddot{\alpha} = -i\omega\dot{\alpha} - G\alpha \qquad \alpha = e^{\lambda t}\alpha_0$$

$$\alpha_0\lambda^2 = -i\omega\lambda\alpha_0 - G\alpha_0 \Rightarrow \lambda^2 + i\omega\lambda + G = 0$$

$$\lambda_\pm = -\frac{i}{2}\left(\omega \pm \sqrt{4G+\omega^2}\right)$$

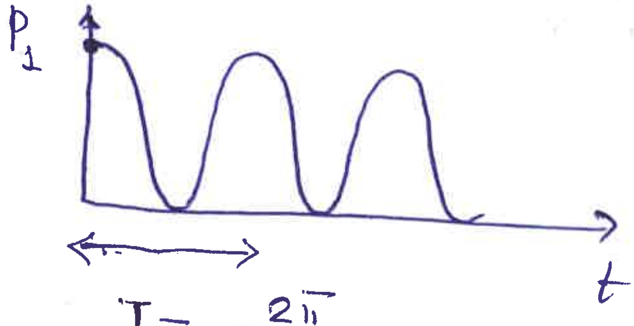
Решение: осцилляции

Вероятность 1 фотона в резонаторе

$$P_1(t) = |\alpha(t)|^2 = \left|\frac{1}{2}\underbrace{\left(e^{-\frac{i}{2}(\omega+\sqrt{4G+\omega^2})t} + e^{-\frac{i}{2}(\omega-\sqrt{4G+\omega^2})t}\right)}_{=1 \quad t=0}\right|^2$$

$$= \cos\left(\sqrt{4G+\omega^2}\,t/2\right)^2$$

Итог: бесконечная память приводит к тому что система никогда не потеряет фотон — он будет вечно возвращаться в систему.



$$T = \frac{2\pi}{\sqrt{G+\omega^2/4}}$$

Что будет, если память конечна?

Ожидается что-то такое

Затухающие колебания

Решим уравнение точно методом Лапласа (метод Лапласа удобнее метода Фурье для конечных времен)

Преобразуем уравнение

$$\dot\alpha = -i\omega\alpha - \int_0^t K(t-T)\,\alpha(T)\,dT \qquad \alpha = e^{-i\omega t}\tilde\alpha$$

$$\dot{\tilde\alpha} = -\int_0^t \tilde K(t-T)\,\tilde\alpha(T)\,dT$$

$$\tilde K(t) = \sum_k |g_k|^2 e^{-i(\omega_k-\omega)t} = \int d\Omega\, \mathcal{J}(\Omega)\, e^{-i\Omega t}$$

$$\underbrace{\mathcal{J}(\Omega) = \sum_k |g_k|^2\,\delta(\Omega-\omega_k+}_{}$$

Частотное представление

$\mathcal{J}(\Omega)$ – частотная зависимость взаимодействия

Преобразование Лапласе

$$f(s) = \int_0^{+\infty} f(t)\,e^{-ts}\,dt$$

$f(t) \Leftrightarrow$	$f(s)$
1	$1/s$
t	$1/s^2$
t^2	$2/s^3$
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
$e^{-i\omega t}$	$\dfrac{1}{s+i\omega}$

Преобразование производной

$$\int_0^{+\infty} \dot{f}(t) e^{-st} dt = s f(s) - f(t=0)$$

Преобразование интеграла "свертки"

$$\int_0^{+\infty}\left[\int_0^t K(t-T) f(T) dT\right] e^{-st} dt = k(s) f(s)$$

Преобразование уравнения

$$\int_0^{+\infty} e^{-st}\left[\dot{\alpha} = -\int_{t_0}^{t} K(t-T) f(T) dT\right] dT \Rightarrow$$

$$\Rightarrow s\alpha(s) - \underbrace{\alpha(t=0)} = -K(s)\alpha(s)$$

Решение $\alpha(t=0)=1$ – начальное условие

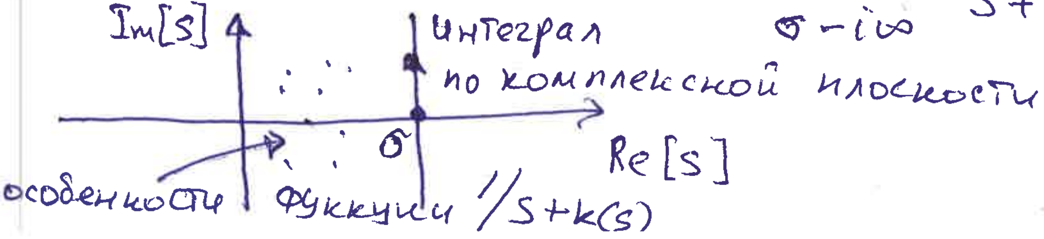
$$\alpha(s) = \frac{1}{s+K(s)}$$

Преобразование ядра

$$K(t) = \int d\Omega\, J(\Omega) e^{-i\Omega t} \Rightarrow \int d\Omega \frac{J(\Omega)}{s+i\Omega}$$

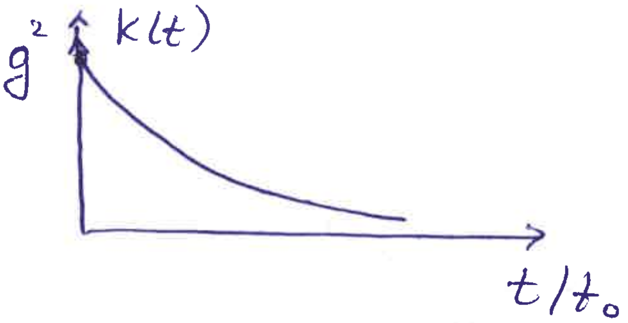
Чтобы найти временную зависимость надо сделать обратное преобразование лапласа

$$\alpha(s) = \frac{1}{s+k(s)} \Rightarrow \alpha(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds \frac{e^{st}}{s+k(s)}$$



Простая модель памяти – экспонента

$$\tilde{K}(t) = g^2 e^{-t/t_0}$$



$$K(s) = \frac{g^2}{s + 1/t_0}$$

частотное представление

$$\mathcal{J}(\Omega) = \mathrm{Re}\left[\frac{g^2}{1/t_0 + i\omega}\right]$$ – функция лоренца

$$\simeq g^2 \frac{t_0}{t_0^2 \omega^2 + 1}$$

Решение

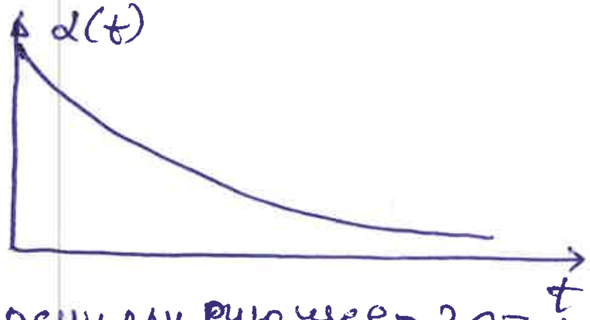
$$d(s) = \frac{1}{s + \frac{g^2}{s + 1/t_0}} \Rightarrow d(t) = e^{-\frac{1}{2}t/t_0}\left[ch\left(\frac{\lambda t}{2}\right) + \frac{sh\left(\frac{\lambda t}{2}\right)}{\lambda t_0}\right]$$

$$\lambda = \sqrt{1/t_0^2 - 4g^2}$$

Два вида решения

а) затухающее решение

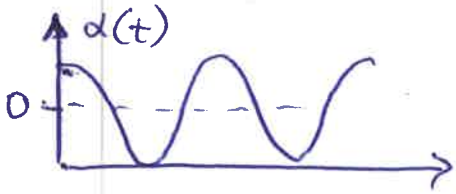
$$\frac{1}{t_0^2} > 4g^2 \Rightarrow \boxed{g t_0 < 1/2} \qquad d(t) \underset{t/t_0 \gg 1}{\simeq} 2 e^{-g^2 t t_0}$$



слабая связь $\boxed{g < \frac{1}{2t_0}}$

в) осциллирующее-затухающее решение

$$\frac{1}{t_0^2} < 4g^2 \Rightarrow \boxed{g t_0 > 1/2} \qquad d(t) \underset{t \gg t_0}{\simeq} e^{-\frac{1}{2}t/t_0} \cos(gt)$$



сильная связь $\boxed{g < \frac{1}{2t_0}}$

Выводы:

a) Потери в квантовуй системе прямо связаны с памятью

в) память зависит от количества и плотности состояний резервуара, чем больше состояний и тем они плотнее, тем быстрее теряется память

с) соотношение как плотности состояний и силы взаимодействия определяют тип временной эволюции: простое затухание (слабая связь) и затухание с осцилляциями (сильная связь)

Приблизетельное вычисление

основное уравнение для матрицы плотности

Рассмотрим ту же модель с гамильтонианом

$$\hat H = \hat H_S + \hat H_R + \hat H_I$$

$$\hat H_S = \hbar\omega\, \hat a^+ \hat a \qquad \hat H_R = \sum_k \hbar\omega_k\, \hat b_k^+ \hat b_k$$

$$\hat H_I = \hbar \sum_k \left(g_k \hat b_k \hat a^+ + g_k^* \hat b_k^+ \hat a\right)$$

1. Перейдем в представление взаимодействия

$$\hat a \to e^{\frac{i\hat H_S t}{\hbar}} \hat a\, e^{-\frac{iHt}{\hbar}} = \hat a\, e^{-i\omega t}$$

$$\hat a^+ \to \qquad \to \hat a^+ e^{i\omega t}$$

$$\hat b_k \to e^{+\frac{i\hat H_R t}{\hbar}} \hat b_k\, e^{-\frac{i\hat H_R t}{\hbar}} = \hat b_k e^{-i\omega_k t}$$

$$\hat b_k^+ \to \qquad \to \hat b_k^+ e^{i\omega_k t}$$

⇒ H_S – не меняется
H_R – не меняется | потому что оно $\hat H_0 = \hat H_S + \hat H_R$

$$\hat H_I \to \tilde{\hat H}_I(t) = \hbar \sum_k g_k e^{-i(\omega_k-\omega)t} \hat b_k \hat a^+ +$$

$$+ \hbar \sum_k g_k^* e^{i(\omega_k-\omega)t} \hat b_k^+ \hat a$$

Мы будем решать уравнение для матрицы плотности. Матрица плотности тут полная — она включает состояния системы + резервуара

Это уравнения Лиувилля- фон Неймана

$$i\hbar \frac{d}{dt}\hat{\rho} = [\hat{H}, \hat{\rho}]$$

Но нам не нужна та часть матрицы плотности, которая отвечает за резервуар

Общее выражение матрицы плотности

$$\hat{\rho} = \sum_{\substack{n_R n_R' \\ n_s n_s'}} |n_R; n_s\rangle \rho^{n_s n_s'}_{n_R n_R'} \langle n_R' n_s'|$$

$$|n_R n_S\rangle = |n_S\rangle \otimes |n_R\rangle$$

Если мы вычисляем любую величину, связанную с состоянием системы, соответствующую ~~величины~~ оператору $\hat{A}$, то мы должны делать так

$$\langle\hat{A}\rangle = Tr[\hat{\rho}\hat{A}] = \sum_{m_R m_S} \langle m_R| \otimes \langle m_S| \hat{\rho}\hat{A} |m_S\rangle \otimes |m$$

$$= \sum_{m_S} \langle m_S| \underbrace{\left[\sum_{m_R} \langle m_R|\hat{\rho}|m_R\rangle\right]}_{} \hat{A} |m_S\rangle =$$

не зависит от состояний $|m_R\rangle$

$\equiv Tr_R[\hat{\rho}] \equiv \hat{\rho}_S$ – "приведенная" матрица плотности

$$= \sum_{m_S} \langle m_S| \hat{\rho}_S \hat{A} |m_S\rangle \equiv Tr_S[\hat{\rho}_S \hat{A}]$$

Таким образом, нам нужна приведенная матрица плотности, а не матрица плотности но уравнение для матрицы $\hat{\rho}_s$ не совпадает с простым уравнением Лиувилля-фон Неймана

$$\underbrace{i\hbar\frac{d}{dt}\hat{\rho}_s=[\hat{H}_s,\hat{\rho}_s]}_{\substack{\text{старое уравнение}\\ \text{Гамильтонова}\\ \text{динамика, законы}\\ \text{сохранения}}}+\underbrace{\hat{\alpha}[\hat{\rho}_s]}_{\substack{\text{добавочные члены}\\ \text{нарушение законов}\\ \text{сохранения}}}$$

— "основное" уравнение

<u>Задача</u>: найти форму основного уравнения — вид оператора $\hat{\alpha}[\hat{\rho}_s]$

Решение методами теории возмущений?

Допущение: нет взаимодействия между S и R до $t_0=0$ (при $t<t_0$)

Тогда матрица плотности до этого момента дается произведением

$$\hat{\rho}(0)=\underbrace{\hat{\rho}_R(0)}_{\substack{\text{матрица}\\ \text{плотности резервуара}}}\underset{\substack{\uparrow\\ \text{матрицы действуют в разных пространствах}\\ \text{состояний}}}{\otimes}\underbrace{\hat{\rho}_S(0)}_{\substack{\text{матрица плотности}\\ \text{системы}}}$$

$\hat{\rho}_R(0)$ — температурное распределение

$$\hat{\rho}_R(0) = \frac{1}{Z}\exp\left[-\frac{\hat{H}_R}{T}\right]$$

Для невзаимодействующих фотонов

$$\hat{\rho}_R(0) = \prod_{n=1}^{+\infty}\frac{1}{Z_n}\underbrace{\exp\left[-\frac{\hbar\omega_n\hat{a}_n^+\hat{a}_n}{T}\right]}$$

$$\langle n'|\exp\left[-\frac{\hbar\omega_n\hat{a}_n^+\hat{a}_n}{T}\right]|n\rangle = e^{-\frac{\hbar\omega n}{T}}\delta_{nn'}$$

Уравнение Лиувилля — фон Неймана в представлени взаимодействия

$\hat{V} = \hat{\tilde{H}}_I$ (переобозначение)

$$i\hbar\frac{d}{dt}\hat{\rho} = \left[H_I(t), \hat{\rho}\right] \Rightarrow \frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}\left[\hat{V}(t), \hat{\rho}\right]$$

$\hat{\rho}$ — полная матрица для S+R.

Решаем методом теории возмущений

$$\hat{\rho}(t) = \hat{\rho}(0) - \frac{i}{\hbar}\int_0^t dt_1\left[\hat{V}(t_1), \hat{\rho}(0)\right] + \left(-\frac{i}{\hbar}\right)^2 \times$$

$$\times\int_0^t dt_1\int_0^{t_1}dt_2\left[\hat{V}(t_1), \left[\hat{V}(t_2), \hat{\rho}(0)\right]\right] + \ldots\ldots$$

Применим операцию редуцирования (приведения)

$$\hat{\rho}_S(t) = \mathrm{Tr}_R\left[\hat{\rho}(t)\right] = \rho_S(0) - \frac{i}{\hbar}\int_0^t dt_1\,\mathrm{Tr}_R\left[\hat{V}(t_1), \hat{\rho}_R(0)\otimes\hat{\rho}_S(0)\right]$$

$$+\left(-\frac{i}{\hbar}\right)^2\int_0^t dt_1\int_0^{t_1}dt_2\,\mathrm{Tr}_R\left\{\left[\hat{V}(t_1), \left[\hat{V}(t_2), \hat{\rho}_R^{(0)}\otimes\hat{\rho}_S(0)\right]\right]\right\} + \ldots$$

Запишем это в "абстрактном" виде

$$\hat\rho_s(t) = \left[1 + \hat U_1 + \hat U_2 + \dots\right]\hat\rho_s(0)$$

$$\hat\rho_s(t) = \left[1 + \hat U_1 + \hat U_2 + \dots\right]\underbrace{\hat U^{-1}(t)\,\hat\rho_s(t)}_{\hat\rho_s(0)}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{d}{dt}\hat\rho_s = \left[\dot{\hat U}_1 + \dot{\hat U}_2 + \dots\right]\hat U^{-1}(t)\,\hat\rho_s(t)$$

Рассмотрим такие взаимодействия, которые удовлетворяют условию $\mathrm{Tr}\left[\hat V\,\hat\rho_R(0)\right] = 0$

Тогда первый член ~~теории~~ ряда теории возмущений равен "0"!

$$\hat U = 1 + \hat U_2 + \dots$$

$$\dot{\hat U}_2\,\hat\rho_s(0) = \left(-\frac{i}{\hbar}\right)^2\int_0^t dt_2\,\mathrm{Tr}_R\left\{\left[\hat V(t), \left[\hat V(t_2), \hat\rho_R(0)\otimes\hat\rho_s(0)\right]\right]\right\}$$

$$\hat U^{-1}(t) = \left[\hat 1 + \underbrace{\hat U_1}_{=0} + \hat U_2 + \dots\right]^{-1} \simeq \hat 1 \quad (\text{приближение})$$

Итак, получаем формально следующее уравнение

$$\frac{d}{dt}\hat\rho_s \simeq \hat{\mathcal{L}}(\cancel{t})\left[\hat\rho_s(t)\right] = \dot{\hat U}_2\,\hat\rho_s(t) =$$

$$= \int_0^t dt_2\,\mathrm{Tr}_R\left\{\left[\hat V(t), \left[\hat V(t_2), \hat\rho_R(0)\otimes\rho_s(t)\right]\right]\right\}$$

~~[illegible]~~

= ~~[illegible]~~

Вычисление $Tr_R\{\cdot\}$

$$V(t) = V_1(t) + V_2(t) = \hbar \sum_k g_k \hat{a}^+ \hat{b}_k e^{-i(\omega_k - \omega)t} +$$

$$+ \hbar \sum_k g_k^* \hat{b}_k^+ \hat{a}\, e^{i(\omega_k - \omega)t}$$

$$[\hat{V}(t), [\hat{V}(t_2), \underbrace{\hat{\rho}_R(0) \otimes \hat{\rho}_S(t)}_{\hat{R}}]] = [V(t), (\hat{V}(t_2)\hat{R} - \hat{R}\hat{V}(t_2))] =$$

$$= \hat{V}(t)\hat{V}(t_2)\hat{R} - V(t)R\hat{V}(t_2) - R\hat{V}(t_2)\hat{V}(t) + \hat{R}\hat{V}(t_2)\hat{V}(t$$

$$Tr_R\{(\hat{V}_1(t) + \hat{V}_2(t))(\hat{V}_1(t_2) + \hat{V}_2(t_2))\hat{R}\} = Tr_R\{V_1(t)V_1(t_2)\hat{R}$$

$$+ \hat{V}_2(t)\hat{V}_2(t_2)\hat{R} + \hat{V}_1(t)\hat{V}_2(t_2)R + V_2(t)V_1(t_2)R\}$$

$$Tr_R\{\hat{V}_1(t)\hat{V}_1(t_2)\hat{R}\} = \sum_{kk'} \hbar^2 g_k^2 \hat{a}^+\hat{a}^+ \underbrace{\langle \hat{b}_k \hat{b}_{k'} \rangle}_{} e^{-i[(\omega_k - \omega)t + i(\omega_{k'}} \rho_S($$

$$= Tr_R[\hat{b}_k \hat{b}_{k'} \hat{\rho}_R(0)] = 0$$

$$Tr_R\{\hat{V}_2(t)\hat{V}_2(t_2)\hat{R}\} = 0$$

$$Tr_R\{\hat{V}_2(t)\hat{V}_1(t_2)\hat{R}\} = \sum_{kk'} \hbar^2 |g_k|^2 \hat{a}\hat{a}^+ \underbrace{\langle \hat{b}_k^+ \hat{b}_{k'} \rangle}_{} e^{-i(\omega_k - \omega)t + i(\omega_k - \omega)} \rho_S(t)$$

$$Tr_R[\hat{b}_k^+ \hat{b}_{k'} \hat{\rho}_R(0)] = n_k \delta_{kk'}$$

$$n_k = \frac{1}{e^{\hbar\omega_k/T} - 1}$$

$$Tr_R\{\hat{V}_1(t)\hat{V}_2(t_2)\hat{R}\} = \sum_{kk'} \hbar^2 |g_k|^2 \hat{a}^+\hat{a} \underbrace{\langle \hat{b}_k \hat{b}_{k'}^+ \rangle}_{(1+n_k)\delta_{kk'}} e^{i(\omega_k - \omega)t - i(\omega_k} \rho_S(t)$$

Кроме того $\langle \hat{b}_k \rangle = \langle \hat{b}_k^+ \rangle = 0$ $Tr_R\{\hat{V}\hat{\rho}_R(0)\} = 0$ удовлетворяется

Вычисляем все члены и получаем

$$\frac{d}{dt}\hat{\rho}_s = \Gamma_1\left[2\hat{a}\hat{\rho}_s\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho}_s - \hat{\rho}_s\hat{a}^+\hat{a}\right] +$$

$$\Gamma_2\left[2\hat{a}^+\hat{\rho}_s\hat{a} - \hat{a}\hat{a}^+\hat{\rho}_s - \hat{\rho}_s\hat{a}\hat{a}^+\right]$$

$$\Gamma_1 = \sum_k |g_k|^2(n_k+1)\int_0^t dt_2\, e^{-i(\omega_k-\omega)(t_2-t)}$$

$$\Gamma_2 = \sum_k |g_k|\, n_k \int_0^t dt_2\, e^{i(\omega_k-\omega)(t_2-t)}$$

$$\int_0^t dt_2\, e^{i\Omega(t_2-t)} \equiv \int_0^t d\tau\, e^{-i\Omega\tau} \underset{t\to\infty}{\simeq} 2\pi\delta(\Omega)$$

(та же аппроксимация)

$$\Rightarrow \Gamma_1 = \sum_k |g_k|^2(1+n_k)\, 2\pi\delta(\omega-\omega_k)$$

$$\Gamma_2 = \sum_k |g_k|^2 n_k\, 2\pi\delta(\omega-\omega_k)$$

$n_k \equiv n_\omega$ (зависит только от энергии) $= \dfrac{1}{e^{\frac{\hbar\omega_k}{T}}-1}$

$$\Gamma_1 = \frac{\gamma}{2}(1+n_\omega) \qquad \Gamma_2 = \frac{\gamma}{2}n_\omega$$

[illegible] $\dfrac{\gamma}{2} = 2\pi\sum\limits_k |g_k|^2\delta(\omega-\omega_k)$ –

вероятность перехода в золотом правиле Ферми

Итак запишем основное уравнение с дополнительным гамильтонианом $\hat{\tilde{H}}_s$ как

$$\frac{d}{dt}\hat{\rho}_s = -\frac{i}{\hbar}\left[\hat{\tilde{H}}, \hat{\rho}_s\right] + \frac{\gamma}{2}(1+n_\omega)\left[2\hat{a}\tilde{\rho}_s\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho}_s - \hat{\rho}_s\hat{a}^+\hat{a}\right.$$

$$+ \frac{\gamma}{2} n_\omega \left[2\hat{a}^+\hat{\rho}_s\hat{a} - \hat{a}\hat{a}^+\hat{\rho}_s - \hat{\rho}_s\hat{a}\hat{a}^+\right]$$

Это уравнение <u>не</u> сохраняет число частиц (фотонов) в резонаторе и их энергию!

Пример: рассмотрим изменение среднего числа фотонов в системе

$$\hat{N} = \hat{a}^+\hat{a}$$

$$\langle N\rangle = \mathrm{Tr}\left[\hat{a}^+\hat{a}\,\hat{\rho}_s\right]$$

$$\frac{d}{dt}\langle N\rangle = \mathrm{Tr}\left[\hat{a}^+\hat{a}\,\dot{\hat{\rho}}_s\right] = -\gamma\langle N\rangle + \gamma n_\omega$$

Решение:

$$\langle N\rangle = N_0 e^{-\gamma t} + n_\omega(1 - e^{-\gamma t})$$

N_0 – начальное значение числа

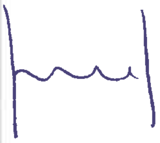
$\Rightarrow$ $\langle N\rangle \underset{t\to\infty}{\Rightarrow} n_\omega$ – температурное распределен

$$\langle N\rangle = \frac{1}{e^{\omega\hbar/T} - 1}$$

Таким образом, в конце концов устанавливается тепловое равновесие

VII. Dynamics with losses
(time evolution of the density matrix)

Simple example: cavity modes



mirrors are not idea => losses of photons

QM — open system

Hamiltonian $H = H_S + H_R + H_I$

Example: single mode in the resonator is coupled to e-m modes outside resonator.

Important assumptions: 1) the mode outside resonator are continuous

2) the mode outside resonator are thermaliz

$H_S = \hbar \omega a^+ a$

$H_R = \sum_k \hbar \omega_k b_k^+ b_k$

$H_I = V = \hbar \sum_k g_k b_k a^+ + \hbar \sum_k g_k^* b_k^+ a$

- **Equation of motion for density matrix**

which density matrix? density matrix of the subsystem of interest (resonator mode, quantum system etc)

Statistical operator

$\hat{\rho}_S = \sum_j p_{\beta_j} |\beta_j\rangle\langle\beta_j|$ $\qquad p_{\beta_j}$ - statistical weigths.

$|\beta_j\rangle$ - orthogonal normal states

Consider operators $\hat{P}_{\nu\nu'} = |\nu\rangle\langle\nu'|$

$$\rho_{\nu\nu'} = \langle \hat{P}_{\nu\nu'} \rangle = Tr(\hat{\rho}_S \hat{P}_{\nu\nu'}) = \langle\nu|\hat{\rho}_S|\nu'\rangle$$

$\Rightarrow$ Statistical operator in states $|\nu\rangle$

$$\hat{\rho}_S = \sum_{\nu\nu'} \rho_{\nu\nu'} |\nu\rangle\langle\nu'|$$

- **Important property**

define $\vec{\beta}_\nu = \begin{pmatrix} \sqrt{p_1}\,\langle\nu|\beta_1\rangle \\ \sqrt{p_2}\,\langle\nu|\beta_2\rangle \\ \vdots \end{pmatrix}$

$$(\vec{\beta}_\nu^* \cdot \vec{\beta}_{\nu'}) = \sum_j p_{\beta_j} \langle\nu'|\beta_j\rangle\langle\beta_j|\nu\rangle = \rho_{\nu\nu'}$$

$$\Rightarrow |\rho_{\nu\nu'}|^2 = |\vec{\beta}_\nu^* \cdot \vec{\beta}_{\nu'}|^2 \leq |\beta_\nu|^2 |\beta_{\nu'}|^2 = \rho_{\nu\nu}\rho_{\nu'\nu'}$$

Schwarz inequality

Schrödinger equation for $|\nu\rangle$

$$i\hbar \frac{\partial}{\partial t}|\nu\rangle = \hat{H}|\nu\rangle$$

$$-i\hbar \frac{\partial}{\partial t}\langle\nu| = \langle\nu|\hat{H}^{+} \qquad \hat{H} = \hat{H}^{+}$$

$$\hat{\rho} = \sum_{\nu\nu'} \rho_{\nu\nu'}|\nu\rangle\langle\nu'|$$

$$i\hbar\frac{\partial}{\partial t}\rho = \sum_{\nu\nu'}\rho_{\nu\nu'}\left[\left\{i\hbar\frac{\partial}{\partial t}|\nu\rangle\right\}\langle\nu'| + i\hbar|\nu\rangle\frac{\partial}{\partial t}\langle\nu'|\right] =$$

$$= \sum_{\nu\nu'}\rho_{\nu\nu'}\left\{\left(\hat{H}|\nu\rangle\right)\langle\nu'| - |\nu\rangle\left(\langle\nu'|\hat{H}\right)\right\} =$$

$$= \hat{H}\hat{\rho} - \rho\hat{H} = [\hat{H},\hat{\rho}]$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\hat{\rho} = -\frac{i}{\hbar}[H,\rho]}$$ — Liouville equation

Consider a single mode coupled to a bath

$$\hat{H} = \hat{H}_S + \hat{H}_I \qquad \hat{\varrho}_S$$

$$\frac{\partial}{\partial t}\hat{\rho}_S = \underbrace{-\frac{i}{\hbar}[\hat{H}_S,\hat{\rho}_S]}_{\substack{\text{quantum dynamics} \\ \text{coherant dynamis} \\ \text{Hamiltonian dynamics}}} + \underbrace{\hat{L}(\rho_S)}_{\substack{\text{incoherent} \\ \text{dynamis} \\ \text{Losses, etc}}}.$$

Example: coherent dynamis

Consider $\hat{\rho}_{nn} = |n\rangle\langle n|$

=> this is diagonal element of operator $\hat{N} = \hat{a}^{+}\hat{a}$

$$\frac{d}{dt}\hat{\rho}_{nn} = -\frac{i}{\hbar}\left[\hat{H}_S ; \hat{\rho}_{nn}\right] = $$

$$= -\frac{i\hbar\omega}{\hbar}\left[a^{+}a|n\rangle\langle n| - |n\rangle\langle n|a^{+}a\right] = 0.$$

Fock space diagonalises the Hamiltonian.

How to take into account losses?

1) Master equations
2) Phenomenological approach
3) Models of quantum decoherence

1) Master equation.

$$H = H_S + H_R + H_I$$

a) write everything in the interaction representation

$H_S = \omega\hbar a^{+}a$

$a \to a(t) \to a e^{-i\omega t}$

$a^{+} \to a^{+}(t) \to a^{+}e^{i\omega t}$

$H_R = \sum_k \hbar\omega_k b_k^{+} b_k$

$b_k \to b_k(t) \to b_k e^{-i\omega_k t}$

$b_k^{+} \to b_k^{+}(t) \to b_k^{+} e^{i\omega_k t}$

$$H_I \Rightarrow \hbar \sum_k g_k b_k a^+ + \hbar \sum_k g_k^* b_k^+ \hat{a} \rightarrow$$

$$\rightarrow \hbar \sum_k g_k e^{-i(\omega_k-\omega)t} \hat{b}_k \hat{a}^+ + \hbar \sum_k g_k^* e^{i(\omega_k-\omega)t} \hat{b}_k^+ \hat{a}$$

Important we consider density matrix for all states resonator and scattering states

$\hat{\rho}^{T}$

But (!) we need only the part of it, that corresponds to resonator states. This part is obtained by tracing out all other degrees of freedom

$$\hat{\rho}_S = \mathrm{Tr}_R[\hat{\rho}^{T}] = \sum_{\{n_R\}} \rho^{n_s n'_s}_{n_R n_R}$$

Assumption: there is no interaction at some point $t=0$ (one can also assume that there is no interaction at $t \to -\infty$), $V(t) \rightarrow$

$$\Rightarrow \hat{\rho}(0) = \hat{\rho}_R(0) \otimes \hat{\rho}_S(0)$$

$\hat{\rho}_R(0)$ – is given by temperature distribution

$$\hat{P}_R(0) = \frac{1}{Z} \exp\left[-\frac{\hat{H}_R}{T}\right]$$

Solve Liouville equation perturbatively

$$\frac{\partial \hat\rho}{\partial t} = -\frac{i}{\hbar}\left[\hat V(t), \hat\rho\right] \qquad \hat V(t) \text{ is small}$$

$$\hat\rho(t) = \hat\rho(0) + \left(-\frac{i}{\hbar}\right)\int_0^t dt_1 \left[\hat V(t_1), \hat\rho(0)\right] + \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_2} dt_2 \, \times$$

$$\times \left[\hat V(t_1), \left[\hat V(t_2), \hat\rho(0)\right]\right] + \ldots$$

Calculate reduced density matrix

$$\hat\rho_S(t) = \mathrm{Tr}_R\left\{\hat\rho(t)\right\} = \rho_S(0) + \left(-\frac{i}{\hbar}\right)\int_0^t dt_1 \, \mathrm{Tr}_R\left\{\left[\hat V(t_1), \hat\rho_R \otimes\right.\right.$$

$$+ \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, \mathrm{Tr}_R\left\{\left[\hat V(t_1), \left[\hat V(t_2), \hat\rho_R \otimes \rho_S(0)\right]\right]\right\} + \ldots$$

$$\hat\rho_S(t) = \left(1 + \hat U_1 + \hat U_2 + \hat U_3 \ldots\right) \hat\rho_S(0)$$

$$\hat\rho_S(t) = \left(1 + \hat U_1 + \hat U_2 + \ldots\right) \hat U^{-1}(t) \hat\rho_S(t)$$

$$\frac{\partial}{\partial t}\hat\rho_S(t) = \left(\dot{\hat U}_1 + \dot{\hat U}_2 + \ldots\right) \hat U^{-1} \hat\rho_S(t)$$

Consider such interaction that $\mathrm{Tr}\,[\hat V \rho_R] = 0$.

$$\hat U \approx \hat 1 + \hat U_2$$

$$\hat U_2 = \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt_2 \, \mathrm{Tr}_R\left\{\left[\hat V(t), \left[\hat V(t_2), \rho_R \otimes (\cdot)\right]\right]\right\}$$

$$\hat U^{-1} = \frac{1}{1 + U_1 + U_2 + \ldots} \approx \hat 1$$

$\Rightarrow$ Simplified Liouvill equation for the reduce density matrix

$$\boxed{\frac{\partial}{\partial t}\hat{\rho}_S(t) = \hat{L}(t)\,\hat{\rho}_S(t)}$$

Calculation of the operator $\hat{L}(t)$

$$\mathrm{Tr}_R\{[\hat{V}(t),[V(t_2), \rho_R \otimes (\cdot)]]\}$$

· ~~Main assumption averages do not~~

$$\hat{V}(t) = \hbar \sum_k g_k \hat{b}_k \hat{a}^+ e^{-i(\omega_k - \omega)t} + \hbar \sum_k g_k^* \hat{b}_k^+ \hat{a} e^{i(\omega_k - \omega)t}$$

$$\mathrm{Tr}_R[\hat{b}_k^+ \hat{b}_{k'} \hat{\rho}_R] = \langle \hat{b}_k^+ \hat{b}_{k'} \rangle = n_k \delta_{kk'}$$

$$n_k = \frac{1}{e^{\varepsilon_k/T} - 1}$$ Bose distribution

$$\langle \hat{b}_k^+ b_{k'}^+ \rangle = \langle \hat{b}_k \hat{b}_{k'} \rangle = 0.$$

· Note $\langle b_k \rangle = \langle b_k^+ \rangle = 0$ $\quad (\mathrm{Tr}_R\{\hat{V}\rho_R\} = 0)$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} = \frac{\gamma}{2}(\bar{n}_R + 1)(2\hat{a}\hat{\rho}a^+ - a^+a\rho - \rho a^+ a) + \frac{\gamma}{2}\bar{n}_R(2a^+\rho a - aa^+\rho - \rho a a^+)$$

master equation

$$\frac{\partial}{\partial t}\rho = \Gamma_1(2a\rho a^+ - a^+a\rho - \rho a^+a)$$
$$+\Gamma_2(2a^+\rho a - aa^+\rho - \rho aa^+)$$

$$\rho = \sum_{n,m} \rho_{nm} |n\rangle\langle m|$$

$$a|n\rangle\langle m|a^+ = \sqrt{nm}\,|n-1\rangle\langle m-1|$$

$$a^+a|n\rangle\langle m| = n|n\rangle\langle m|$$

$$|n\rangle\langle m|a^+a = |n\rangle\langle m|m$$

$$a^+|n\rangle\langle m|a = |n+1\rangle\langle m+1|\sqrt{(n+1)(m+1)}$$

$$aa^+|n\rangle\langle m| = |n\rangle\langle m|(n+1)$$

$$|n\rangle\langle m|aa^+ = |n\rangle\langle m|(m+1)$$

$$\sum \dot{\rho}_{nm}|n\rangle\langle m| = \Gamma_1 \sum \rho_{nm}\,(2\sqrt{mm}\,|n-1\rangle\langle m-1|$$
$$-|n\rangle\langle m|\,n - m$$

Correction

befor lecture

$H = H_M + H_F + H_{MF}$ $\qquad$ $g\sqrt{n-1} = g\sqrt{n}\left(1-\frac{1}{2n}\right)$

$H_M = |A\rangle\langle A|\, \omega_A \hbar$

$H_{MI} = \hbar\, |A\rangle\langle G|\, f e^{+i\omega_L t} + |G\rangle\langle A|\, f e^{-i\omega_L t}$

$H_A = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\hbar\omega_A + \underline{\begin{pmatrix}0 & fe^{+i\omega t}\\ fe^{-i\omega t} & 0\end{pmatrix}}$ $\quad \sim n \to \infty$

$E_n^\pm - E_{n-1}^\pm = \omega_L \pm g/2\sqrt{n}$

$i\frac{\partial}{\partial t}\hbar|\psi\rangle = \cancel{\hbar}\hat{H}\psi$ $\qquad$ $E_n^\pm - E_{n-1}^\mp = \omega_L \pm 2g\sqrt{n}$

$E =$

$i\frac{\partial}{\partial t}\hbar|\psi\rangle = H_0|\psi\rangle$ $\quad \begin{cases}\omega_L n \pm g\sqrt{n}\\ \omega_L(n-1) \pm g\sqrt{n-1}\end{cases}$

$|\psi\rangle = e^{-\frac{iH_0 t}{\hbar}}|\varphi\rangle$

$\cancel{H_0|\varphi\rangle +} i\hbar\frac{\partial}{\partial t}|\varphi\rangle e^{-\frac{iH_0 t}{\hbar}} = (\cancel{H_0} + H_{MF})|\varphi\rangle e^{-iH_0 t}$

$i\hbar\frac{\partial}{\partial t}|\varphi\rangle = e^{iH_0 t/\hbar} H_{MF} e^{-iH_0 t/\hbar}|\varphi\rangle$

$e^{i\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\omega_A t}\sigma_+ e^{-i\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\omega_A t} = \begin{pmatrix}e^{i\omega_A t} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}e^{-i\omega_A t} & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}e^{i\omega_A t} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}e^{i\omega_A t}$

$\begin{pmatrix}e^{i\omega_A t} & 0\\ 0 & 1\end{pmatrix}\underbrace{\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{-i\omega_A t} & 0\\ 0 & 1\end{pmatrix}}_{\begin{pmatrix}0 & 0\\ e^{-i\omega_A t} & 0\end{pmatrix}} =$

$= \begin{pmatrix}0 & 0\\ e^{-i\omega_A t} & 0\end{pmatrix} = e^{-i\omega_A t}\sigma_-$

$H'_{MI} = e^{iH_0 t/\hbar} H_{MI} e^{-iH_0 t/\hbar} = f\sigma^+ + f^*\sigma^-$

$Z = \mathrm{Tr}\,\rho$

$\dot{Z} = \frac{\gamma}{2}(n_R+1)(2\langle a\rho a^+\rangle - \langle a^+a\rho\rangle - \langle \rho a^+a\rangle)$

$+\frac{\gamma}{2}n_R(2\langle a^+\rho a\rangle - \langle aa^+\rho\rangle - \langle \rho a a^+\rangle)$

$= \frac{\gamma}{2}(1+n_R)\cdot 0 + \frac{\gamma}{2}n_R\cdot 0.$

$N = \langle a^+a\rangle$

$\dot{N} = \frac{\gamma}{2}(n_R+1)(2\langle a^+a\,a\rho a^+\rangle - \langle a^+a\,a^+a\rho\rangle - \langle a^+a\,\rho a^+a\rangle)$

$+\frac{\gamma}{2}n_R(2\langle a^+a\,a^+\rho a\rangle - \langle a^+a\,aa^+\rho\rangle - \langle a^+a\,\rho a a^+$

$\langle a^+a\,a\rho a^+\rangle = \langle a^+\underbrace{a^+a}\,a\rho\rangle = \langle a^+(aa^+-1)\,a\,\rho\rangle = -\,a^+a\rho$

1. $-\frac{\gamma}{2}(n_R+1)\,2N = -\gamma(1+n_R)N$

2. $\langle a^+a\,a^+\rho a\rangle = \langle \underbrace{aa^+a\,a^+}\rho\rangle \;|\; \frac{\gamma}{2}(n_R)\cdot 2\cdot(Z+N)$

$= \gamma\cdot n_R(Z+N)$

$1+2 \Rightarrow -\gamma(1+n_R)N + \gamma n_R Z + \gamma n_R N = -\gamma N + \gamma n_R$

Aufgabe 22, Pionische Atome

Charged pions can be created in high energy proton-proton scattering processes, for example, $p+p \to p+p+\pi^+ + \pi^-$. When so created negatively charged pion collides with an atom it can be captured by the potential of its nuclei thereby emitting one of the atomic electrons. Such atoms with pions substituting electrons are called pionic atoms.

Assume that the Coulomb interaction with the Z-charge of the nuclei is the main binding force for pions in a pionic atom and neglect all other contributions. The quantum pionic field can be described by the Klein-Gordon equation in exercise 21 with $\underline{A} = \underline{0}$ and $qA^0 = -\frac{Ze^2}{4\pi\varepsilon_0 c|\underline{x}|}$. Consider the monochromatic solution in the form $\psi(\underline{x},t) = \varphi(\underline{x})\,e^{-i\omega t}$ and calculate possible values of $\hbar\omega$ $(\omega > 0)$ for the states that can be normalized. Express $\hbar\omega$ via the fine structure constant $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$. Discuss, what takes place for $Z > \frac{1}{2\alpha}$ and investigate qualitatively for which Z this is the case. Derive the expansion for the obtained $\hbar\omega$ with respect to α including terms of the order of $\mathcal{O}(\alpha^4)$ and compare the result with the energy levels of the hydrogen atom in the non-relativistic quantum theory.

$aa^+ - a^+a = 1$

$\langle a^+a\,aa^+\rho\rangle + \langle a^+a\,\rho\,aa^+\rangle = \langle a^+a\,aa^+\rho\rangle + \langle a^+a^+a\,a^+a$

aa^+aa^+

$$\mathring{p} = \frac{\gamma}{2}(1+n_R)(2apa^+ - a^+\overset{a}{p}\cancel{a} - pa^+a)$$

$$+\frac{\gamma}{2} n_R(2a^+pa - aa^+p - paa^+)$$

$$p = \sum_i p_{nm}|n\rangle\langle n|$$

$$a|n\rangle\langle n|a^+ = \cancel{\sqrt{\ }}\, n|n-1\rangle\langle n-1|$$

$$a^+a|n\rangle\langle n| = n|n\rangle\langle n|$$

$$|n\rangle\langle n|a^+a = |n\rangle\langle n|\, n$$

$$\frac{\gamma}{2}(1+n_R)(2p_{n+1}(n+1) - 2p_n n) +$$

$$\frac{\gamma}{2} n_R(2p_{n-1}\cdot n - 2p_n(n+1)) = 0$$

$$(1+n_R)(p_{n+1}$$

$$a^+|n\rangle\langle n|a = (n+1)\, |n+1\rangle\langle n+1|$$

$$aa^+|n\rangle\langle n|$$

Aufgabe 22, Pionische Atome

Geladene Pionen können in hochenergetischen Proton-Proton Streuprozessen erzeugt werden, z.B. in einem Prozess $p+p \to p+p+\pi^+ + \pi^-$. Fliegen die hierbei erzeugten negativ geladenen Pionen π^- an einem Atom vorbei, kann es vorkommen, dass sie im Potential des Atomkerns eingefangen werden bei gleichzeitiger Emission eines zuvor zum Atom gehörigen Elektrons. Die so entstandenen gebundenen Zustände nennt man pionische Atome.

Nehmen Sie an, dass die Coulomb Wechselwirkung mit dem Z-fach geladenen Kern die Hauptursache für die Bindung der Pionen in pionischen Atomen ist und vernachläßigen Sie alle weiteren Beiträge, z.B. aufgrund der starken Wechelwikung mit dem Kern. Das Pion-Feld im Einfluss des Coulomb-Potentials des Kerns kann dann durch die in Aufgabe 21 hergeleitete Klein-Gordon-Gleichung mit $\underline{A} = \underline{0}$ und $qA^0 = -\frac{Z e^2}{4\pi\varepsilon_0 c|\underline{x}|}$ beschrieben werden. Machen Sie einen monochromatischen Lösungsansatz der Form: $\psi(\underline{x}, t) = \varphi(\underline{x})\, e^{-i\omega t}$ und bestimmen Sie für normierbare Zustände die möglichen Werte für $\hbar\omega$, für die $\omega > 0$ ist. Drücken Sie $\hbar\omega$ durch die Feinstrukturkonstante $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$ aus. Diskutieren Sie, was für $Z > \frac{1}{2\alpha}$ passiert und ermitteln Sie quantitativ für welche Z dies der Fall ist. Entwickeln Sie die für $\hbar\omega$ gefundenen Werte nach α bis zu Termen der Ordnung $\mathcal{O}(\alpha^4)$ einschließlich und vergleichen Sie das Ergebnis mit dem in der nicht-relativistischen Quantentheorie bestimmten Ausdruck für die Energie des Wasserstoff Atoms.

$$\langle n|a^+a = (a|n\rangle)^+ = \sqrt{n}\, \langle n|$$

$$\langle n-1|a = (a^+|n\rangle)^+ = \sqrt{n}$$

$$(1+n_R)(p_{n+1}(n+1) - p_n n) \pm n_R(p_n\cdot\cancel{n} - p_{n-1}\, n).$$

$$(p_{n+1})(1+n_R) = n_R\, p_n$$

$(n+1)$

$$\boxed{\frac{p_{n+1}}{p_n} = \frac{n_R}{n_R+1}}$$

- Lindblad phenomenological approach

$$\dot{\rho}_S = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}_S] + \hat{\mathcal{D}}[\rho_S]$$

Can ~~b~~ we describe the time evolution due to the interaction with the bath ~~as~~ by choosin a set of phenomenological damping const

Example: if we have oscillatory equation

$$\dot{f} = -\cancel{\frac{i}{\hbar}}\, i\omega f \quad \text{with solution } f = e^{i\omega t} f$$

we can add damping rate Γ

$$\dot{f} = -i\omega f - \Gamma f \Rightarrow f = e^{-i\omega t - \Gamma t} f_0$$

$$\left\{ \begin{array}{l} f = e^{-i\cancel{\omega} t} f_0 \\ \cancel{\omega} = \omega - i\Gamma \end{array} \right\}$$

Can we do the same for the reduced density matrix

Important: 1) one has to satisfy the usual conditions $\mathrm{Tr}_S\, \hat{\rho}_S = \cancel{0}\, 1$

$$\rho_{S\cancel{s}nn'} \geqslant 0 \qquad |\rho_{nn'}|^2 \leq \rho_{nn}\rho_{n'n'}$$

2) the number of constants is equal to ~~N×N~~ $\frac{N(N-1)}{2}$

Recepie from Linblad

$$\hat{\mathcal{D}}[\hat{\rho}] = \sum_{n,m}^{N^2-1} \frac{h_{n,m}}{2} \left(2\hat{L}_n \rho \hat{L}_m^+ - \rho \hat{L}_m^+ \hat{L}_n - \hat{L}_m^+ \hat{L}_n \rho \right)$$

h_{nm} – is a set of phenomenological constants

$\hat{L}_n$ – ~~op~~ ~~matrix~~ operators

⇒ Master equation ~~conserves~~ preserves essential properties of the ~~system~~ desity matrix

~~$\hat{L}_n$ – corresponds to a particular transition~~

when one writes this in the diagonal form

$$\hat{h} \to \begin{pmatrix} \gamma_1 & & \\ & \gamma_2 & \\ & & \ddots \end{pmatrix}$$

$$\hat{\mathcal{D}}[\hat{\rho}] = \sum_j \frac{\gamma_n}{2} \left(2\hat{L}_j \hat{\rho} \hat{L}_j^+ - \hat{\rho} L_j^+ L_j - L_j^+ L_j \hat{\rho} \right)$$

⇒ $\hat{L}_j$ – correspond to ~~transition~~ lossy transitions

γ_j – corresponding rates

In our case

$$\hat{H}_I = \hbar \sum_k g_k e^{-i(\omega_k-\omega)t} b_k a^+ +$$

$$+ \hbar \sum_k g_k^* e^{i(\omega_k-\omega)t} \cancel{a} b_k^+ a$$

we have two operators: $\hat{L}_1 = \hat{a}$ (photon lost)

$\hat{L}_2 = \hat{a}^+$ (photon created)

so $\cancel{2}$ $\hat{D}(\hat{\rho}) = \frac{\Gamma_1}{2}(2a\hat{\rho}a^+ - \hat{\rho}a^+a - a^+a\hat{\rho})$

$$+ \frac{\Gamma_2}{2}(2\hat{a}^+\hat{\rho}\hat{a} - \hat{\rho}aa^+ - aa^+\rho)$$

$$\Gamma_1 = \underbrace{\gamma}_{} \cdot \underbrace{(n+1)}_{}$$

~~probal~~ number of photons in the bath+1
transition
matrix elemen

$$\Gamma_2 = \gamma(n)$$

- Equations for the operators (observables)

(Quantum Langevin equation)

$$\hat{H} = \hbar\omega_0 a^+ a + \sum_k \hbar\omega_k b_k^+ b_k +$$

$$+ \sum_k \hbar\left(g_k a^+ b_k + g_k^* b_k^+ a\right)$$

Heisenberg equation for operators

$$\dot{a} = -\frac{i}{\hbar}[a, H]$$

$$\begin{cases} \dot{\hat{a}} = -i\omega\hat{a} - i\sum_k g_k \hat{b}_k \\ \dot{\hat{b}}_k = -i\omega_k b_k - i g_k^* \hat{a} \end{cases} \quad \checkmark$$

$$b_k(t) = \underbrace{e^{-i\omega_k t} b_k(t=0)}_{\text{free evolution of the R-modes}} - \underbrace{i g_k^* \int_0^t dt' e^{-i\omega_k(t-t')} a(t')}_{\text{driven evolution of R-modes}}$$

$$\dot{\hat{a}} = -i\omega a - i\sum_k g_k e^{-i\omega_k t} b_k(0) -$$

$$-\sum_k |g_k|^2 \int_0^t dt' \, e^{-i\omega_k(t-t')} a(t')$$

Solution of this equation

$a = e^{-i\omega t} A$ (interaction representation)

$$\dot{\hat{A}} = \underbrace{-i\sum_k g_k e^{-i(\omega_k-\omega)t}\hat{b}_k(0)}_{F(t)} - \int_0^t K(t-t')A(t')\,dt'$$

$$K(t-t') = \sum_k |g_k|^2 e^{-i(\omega_k-\omega)(t-t')}$$

$$\boxed{\dot{\hat{A}} = -\int K(t-t')A(t') + F(t).}$$

- Kernel function introduces memory

resonator modes $\underset{t}{\overset{t'}{\rightleftarrows}}$ Reservoir modes

back action

F(t) – is randomised ~~field~~ fluctuations of reservoir modes

analog in classical Langevin equation

=> Quantum Langevin equation

$$\dot{A} = - \int K(t-t')\, A(t')\, dt'$$

$$K(t) \rightarrow \text{const} = \Omega^2$$

$$\dot{A} = -\Omega^2 \int_0^t A(t')\, dt'$$

$$\frac{d^2}{dt^2} A = -\Omega^2 A \Rightarrow A = A_0 e^{i\Omega t} + B_0 e^{-i\Omega t}$$

Phenomenological damping of b_k modes with a constant Γ_k

$$K(t) = \sum_k |g_k|^2 e^{-i(\omega_k - \omega)t \boxed{- \Gamma_k t}}$$

In the limit $\Gamma \rightarrow \infty$ the kernel becomes closer to the δ function → decay

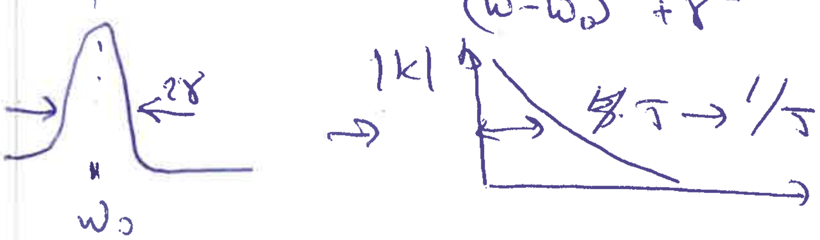
---

$$K(t) = \sum_{k \to \infty} |g_k|^2 e^{-i\omega_k t} \Rightarrow$$

$$\Rightarrow K(t) = \int_{-\infty} \mathcal{J}(\omega)\, e^{-i\omega t} d\omega \qquad \mathcal{J}(\omega) = \sum_k |g_k|^2 \delta(\omega_k - \omega)$$

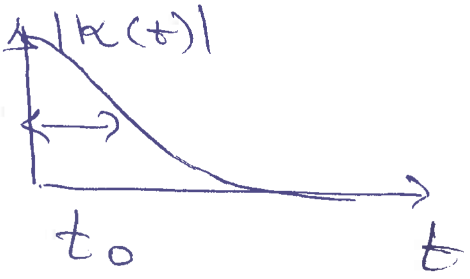
$$\mathcal{J}(\omega) = \frac{\gamma}{(\omega-\omega_0)^2 + \gamma^2}$$

$$\Rightarrow K(t) = \int d\omega \frac{\gamma}{(\omega-\omega_0)^2+\gamma^2} e^{-i\omega t} \rightarrow \underline{e^{-i\omega_0 t - \gamma t}}$$



- Markov approximation (no memory)

We assume that the function $K(t)$ decay fast



t_0 - memory depth length

t_0 is very small

$$\Rightarrow \int_0^t K(t-t')\,a(t')\,dt' \approx a(t)\int_0^t K(t-t')\,dt' \approx$$

$$\approx a(t)\underbrace{\int_0^{+\infty} K(t')\,dt'}_{\Gamma}$$

$$\Rightarrow \underbrace{\dot{A} = -\Gamma A}\, + F(t)$$

$$A = e^{-\Gamma t}A(0)$$

$$\Gamma = \int_0^{+\infty}\sum_k |g_k|^2 e^{-i(\omega_k-\omega)t}dt = \sum_k |g_k|^2$$

$$\int_0^{+\infty} e^{-i\cancel{\omega}\Delta\omega t-\varepsilon t}dt = \cancel{\frac{1}{\omega-\omega_k}}\ \frac{-i}{\Delta\omega \pm i\varepsilon} =$$

$$= -i\frac{1}{\omega_k-\omega-i\varepsilon} = -iP\left(\frac{1}{\omega_k-\omega}\right) + 2\pi\,\delta(\omega_k-\omega)$$

$\gamma \to 0$ $\quad J(\omega) \to \delta(\omega - \omega_0)$ single mode

$$\dot{\alpha} = -i\omega\alpha - f(t) - \int_0^t K(t-t')\,\alpha(t')\,dt'$$

$$k(t) = g\int d\omega\, \delta(\omega-\omega_0)\, e^{-i\omega t} \to |g|^2 e^{-i\omega_0 t}$$

$$\alpha = e^{-i\omega t}\,\tilde{\alpha}$$

$$\dot{\tilde{\alpha}} = -\tilde{f} e^{i\omega t} - |g|^2 \int_0^t e^{-i(\omega_0 - \omega)(t-t')}\,\tilde{\alpha}(t)\,dt$$

Resonance: $\omega = \omega_0$ (frequency of ~~reson~~ system mode = frequency of reservoir)

$$\Rightarrow \quad \dot{\tilde{\alpha}} = \tilde{f}(t) - |g|^2 \int_0^t \tilde{\alpha}(t)\,dt$$

$$\Rightarrow \quad \ddot{\tilde{\alpha}} = \dot{\tilde{f}}(t) - |g|^2 \tilde{\alpha}$$. (driven harmonic oscillator)

$\alpha = e^{i|g|^2 t}$ $\qquad |g|^2$ – Rabi oscillation frequency

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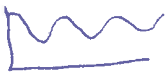
Conclusion:

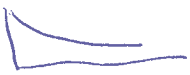
memory defines the type of the ~~relaxation~~ time dynamics

{ long memory → oscillations



non-Markovian process

{ short memory → exponential decay



markov approxim

memory length ↔ width of the coupling spectrum

$$\Gamma = \sum_k |g_k|^2 \left\{ 2\pi\,\delta(\omega_k - \omega) - i P\left(\frac{1}{\omega_k - \omega}\right) \right\}$$

~~g(ω)~~ $G(\omega) = \sum_k |g_k|^2 \delta(\omega_k - \omega)$

$$\Gamma = \gamma - i\Delta$$

$$\underbrace{\gamma = 2\pi\, G(\omega)}_{} \qquad \Delta = ⨍ \frac{G(\omega')}{\omega' - \omega}\, d\omega'$$

perturbation theory transition rate !

Δ – frequency shift

- Exact solution of the Langevine equation (Laplace transforme

$$A(s) = \int_0^{+\infty} A(t)\, e^{-st}\, dt$$

$$\int_0^{+\infty} \dot{A}(t)\, e^{-st}\, dt = A(t)\, e^{-st}\Big|_0^{\infty} + s\int_0^{+\infty} A\, e^{-st}\, dt$$

$$= - A(0) + s A(s)$$

~~[illegible]~~

$$sA(s) - A(0) = - K(s)A(s)\ \hat{F}(s)$$

$$\boxed{A(s) = \frac{A(0) + F(s)}{s + K(s)}}$$

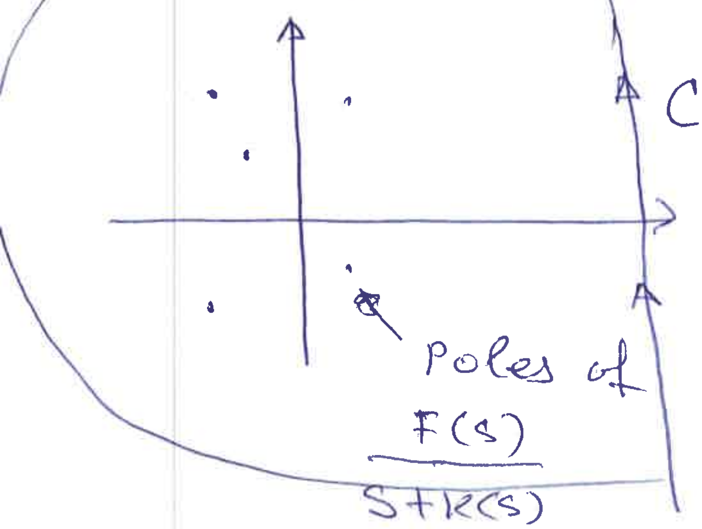
$$\hat{F}(s) = \sum_k \frac{g_k \hat{b}_k(0)}{i(\omega_k - \omega) + s}$$

Inverse Laplace transform

~~⑯~~ ⑰

$$\hat{A}(t) = \frac{1}{2\pi i}\int_C \frac{\hat{A}(0)}{s+k(s)} e^{st} ds + \frac{1}{2\pi i}\int_C \frac{\hat{F}(s)e^{st}}{s+k(s)} ds$$

C – integration contour.



$$H = \hbar\omega a^{+}a + \hbar\omega_k b_k^{+} b_k + \hbar\sum_k g_k b_k^{+} a + c.c.$$

$|\psi\rangle \Rightarrow \quad |1, 0_k\rangle$

$\qquad\qquad |0, 1_k\rangle$

Two-level quantum system in a resovator

$$H = \underbrace{\underbrace{\hbar(\sigma_z + 1)\frac{\omega}{2}}_{H_S} + \underbrace{\omega\hbar a^{+}a}_{H_F} + \underbrace{\hbar g(\sigma^{+}a + a\sigma^{+})}_{H_I}}_{\text{Janes - Cummings model}}$$

We assume photon loss

$L_1 = a$ – Linblad operator

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \frac{\Gamma}{2}(1+\bar{n})(2\hat{a}\hat{\rho}\hat{a}^{+} - \hat{a}^{+}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{+}\hat{a}) + \frac{\Gamma}{2}(\bar{n})(2\hat{a}^{+}\hat{\rho}\hat{a} - \hat{a}\hat{a}^{+}\hat{\rho} - \hat{\rho}\hat{a}^{+}\hat{a}^{+})$$

ρ – reduced density matrix of QS + photons

Consider 3 Lowest states

$|0\rangle = |G_{QS}\rangle|0_{ph}\rangle$; $|1\rangle = |E_{QS}\rangle|0h\rangle$

$|2\rangle = |G_{QS}\rangle|1_{ph}\rangle$.

If a system is initially in state $|2\rangle$ or $|1\rangle$ then only these states are involved in the time evolution ⑲

$$\hat{H}_S|0\rangle = \overset{0}{\cancel{-\frac{\hbar\Omega}{2}|0\rangle}} \qquad \hat{H}_F|0\rangle = 0.$$

$$\hat{H}_S|1\rangle = +\frac{\hbar\Omega}{\cancel{2}}|1\rangle \qquad \hat{H}_F|1\rangle = 0$$

$$\hat{H}_S|2\rangle = \overset{0}{\cancel{-\frac{\hbar\Omega}{2}}}|0\rangle \qquad \hat{H}_F|2\rangle = 0.\hbar\omega$$

$$\hat{H}_S + \hat{H}_F = \sum_{\eta\eta'} |\eta\rangle\langle\eta'| H_{\eta\eta'}$$

$$[\check{H}_{\eta\eta'}] = \hbar \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Omega & 0 \\ 0 & 0 & \omega \end{bmatrix}$$

$$\cancel{\langle 1|H_I|0\rangle = \hbar g\langle 1|\sigma^+ a + a^+\sigma^-|0\rangle}$$

$$\left.\begin{array}{l} \sigma^- a^+|0\rangle = 0 \\ \sigma^+ a|0\rangle = 0 \end{array}\right\} \Rightarrow |0\rangle \text{ uncoupled}$$

$$\left.\begin{array}{l} \sigma^- a^+|1\rangle = |2\rangle \\ \sigma^+ a|1\rangle = 0 \end{array}\right\} \quad \langle 2|H_I|1\rangle = \hbar g$$

$$\check{H}_I = \hbar \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{bmatrix}$$

⇒ DM ~~$\frac{\partial \rho}{\partial t}$~~ Equation for the density matrix

~~$\rho = \Big[\rho_{00}$~~ $\hat{\rho} = \sum_{n'n} \rho_{nn'} |n\rangle\langle n'|$

$$\check{\rho}_{nn'} = \begin{bmatrix} \rho_{00} & \rho_{01} & \rho_{02} \\ \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{20} & \rho_{21} & \rho_{22} \end{bmatrix}$$

Coherent (Hamiltonian dynamics)

$$\frac{\partial}{\partial t} \check{\rho} = -\frac{i}{\hbar} [\check{H}_0, \check{\rho}]$$

$$[\check{H}_0, \check{\rho}] = \Omega \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cancel{1} & 0 \\ 0 & 0 & \cancel{1}0 \end{bmatrix} \begin{bmatrix} \rho_{00} & \rho_{01} & \rho_{02} \\ \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{20} & \rho_{21} & \rho_{22} \end{bmatrix} - \cancel{\Omega} \begin{bmatrix} \rho_{00} & \rho_{01} & \rho_{02} \\ \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{20} & \rho_{21} & \rho_{22} \end{bmatrix} \right\} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \hbar\Omega \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{20} & \rho_{21} & \rho_{22} \end{pmatrix} - \begin{pmatrix} 0 & \rho_{01} & \rho_{02} \\ 0 & \rho_{11} & \rho_{12} \\ 0 & \rho_{21} & \rho_{22} \end{pmatrix} \right\} = \hbar\Omega \begin{pmatrix} 0 & -\rho_{01} & -\rho_{02} \\ \rho_{10} & 0 & 0 \\ \rho_{20} & 0 & 0 \end{pmatrix}$$

$$[\check{H}_I, \check{\rho}] = \hbar g \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} \\ \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{20} & \rho_{21} & \rho_{22} \end{pmatrix} - \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} \\ \rho_{10} & \rho_{11} & \rho_{12} \\ \rho_{20} & \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$$= \hbar g \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \rho_{20} & \rho_{21} & \rho_{22} \\ \rho_{10} & \rho_{11} & \rho_{12} \end{pmatrix} - \begin{pmatrix} 0 & \rho_{02} & \rho_{01} \\ 0 & \rho_{12} & \rho_{11} \\ 0 & \rho_{22} & \rho_{21} \end{pmatrix} \right\} = \hbar g \begin{pmatrix} 0 & -\rho_{02} & -\rho_{01} \\ \rho_{20} & \rho_{21}-\rho_{12} & \rho_{22}-\rho_{11} \\ \rho_{10} & \rho_{11}-\rho_{22} & \rho_{12}-\rho_{21} \end{pmatrix}$$

$$\rho_{00} = 0$$

$$\dot{\rho}_{01} = i\Omega\rho_{01} + ig\rho_{02}$$

$$\dot{\rho}_{02} = i\Omega\rho_{02} + ig\rho_{01}$$

$$\dot{\rho}_{10} = -i\Omega\rho_{10} - ig\rho_{20}$$

$$\dot{\rho}_{20} = -\Omega\rho_{20} - ig\rho_{10}$$

$$\rho_{01} = \rho_{10}^*$$

$$\rho_{02} = \rho_{20}^*$$

$$\dot{\rho}_{11} = gi(\rho_{12} - \rho_{21}) = 2g\,\mathrm{Im}[\rho_{21}]$$

$$\dot{\rho}_{22} = ig(\rho_{21} - \rho_{22}) \approx -2g\,\mathrm{Im}[\rho_{21}]$$

$$\dot{\rho}_{12} = ig(\rho_{11} - \rho_{22})$$

$$\dot{\rho}_{21} = -ig(\rho_{11} - \rho_{22})$$

$$\rho_{12} = \rho_{21}^*$$

$$\dot{\rho}_{11} + \dot{\rho}_{22} = 0 \Rightarrow \rho_{11} + \rho_{22} = 1$$

$$\hat{L} = \hat{a}$$

$$\hat{\mathcal{D}}(\hat{L}) = \frac{\Gamma}{2}(1+\bar{n})\left(2\hat{a}\hat{\rho}\hat{a}^+ - a^+a\hat{\rho} - \hat{\rho}\hat{a}^+\hat{a}\right)$$

$$\hat{\rho} = \sum_{n\nu} \rho_{n\nu}|n\rangle\langle\nu|$$

$\hat{a}|0\rangle = 0$ ~~$a^+|0\rangle = |2\rangle$~~

~~$\langle 0|\hat{a}\hat{\rho}\, a^+|0\rangle = \rho_{22}$~~ ~~$\langle 1|\hat{a}\hat{\rho}\, a^+|1\rangle =$~~

~~$\langle 0|\hat{a}^+\hat{a}\hat{\rho}|0\rangle = 0$~~ ~~$\langle 1|\hat{a}^+\hat{a}\hat{\rho}|1\rangle = 0$~~

~~$\langle 0|\hat{\rho}\hat{a}^+\hat{a}|0\rangle = 0$~~ ~~$\langle 1|\hat{\rho}\hat{a}^+\hat{a}|1\rangle = 0$~~

$\langle 0|\hat{L}|2\rangle = 1 \qquad \langle \eta|\hat{L}|\eta'\rangle = 0$ (others)

$\hat{L} = |0\rangle\langle 2| \qquad L^{+} = |2\rangle\langle 0|$

$$\hat{D}(\hat{L}) = \frac{\Gamma}{2}\underbrace{(1+\bar{n})}_{n=0}\left(2\hat{L}\hat{\rho}\hat{L}^{+} - \hat{L}^{+}\hat{L}\hat{\rho} - \hat{\rho}\hat{L}^{+}\hat{L}\right)$$

$$\hat{L}\hat{\rho}\hat{L}^{+} = |0\rangle\langle 2|\hat{\rho}|2\rangle\langle 0| = |0\rangle \rho_{22}\langle 0|$$

$$\hat{L}^{+}\hat{L}\rho = |2\rangle\langle 0|0\rangle\langle 2|\hat{\rho} = |2\rangle\langle 2|\hat{\rho}$$

$$\hat{\rho}\hat{L}^{+}\hat{L} = \hat{\rho}|2\rangle\langle 0|0\rangle\langle 2| = \hat{\rho}|2\rangle\langle 2|$$

$$\hat{D}(\hat{L}) = \frac{\Gamma}{2}\Big(\overset{2|0\rangle\rho_{22}\langle 0|}{[illegible]} - |2\rangle\langle 2|\hat{\rho} - \hat{\rho}|2\rangle\langle 2|\Big)$$

$\langle 0|\hat{D}(L)|0\rangle = \Gamma\rho_{22}$

$\langle 1|\hat{D}(L)|1\rangle = 0$

$\langle 2|D(L)|2\rangle = -\Gamma\rho_{22}$

$T_1 = \frac{1}{\Gamma}$

$T_2 = \frac{2}{\Gamma} = 2T_1$

$\langle 0|\hat{D}|1\rangle = 0 \qquad \langle 1|\hat{D}|0\rangle = 0$

$\langle 0|\hat{D}|2\rangle = -\frac{\Gamma}{2}\rho_{02} \qquad \langle 2|\hat{D}|0\rangle = -\frac{\Gamma}{2}\rho_{20}$

$\langle 1|\hat{D}|2\rangle = -\frac{\Gamma}{2}\rho_{12} \qquad \langle 2|\hat{D}|1\rangle = -\frac{\Gamma}{2}\rho_{21}$

$$\boxed{\dot{\rho}_{00} = \Gamma\rho_{22}}$$

$$\boxed{\begin{aligned}\dot{\rho}_{01} &= i\Omega\rho_{01} + ig\rho_{02}\\ \dot{\rho}_{02} &= i\Omega\rho_{02} + ig\rho_{01} - \frac{\Gamma}{2}\rho_{02}\end{aligned}}$$

$$\boxed{\begin{aligned}\dot{\rho}_{11} &= 2g\,\mathrm{Im}[\rho_{21}]\\ \dot{\rho}_{22} &= -2g\,\mathrm{Im}[\rho_{21}] - \Gamma\rho_{22}\\ \dot{\rho}_{12} &= ig(\rho_{11}-\rho_{22}) - \frac{\Gamma}{2}\rho_{12}\end{aligned}}$$

$Y_{+} = \rho_{21} - \rho_{12}$

$$\frac{d}{dt}\begin{pmatrix}\rho_{11}\\ \rho_{22}\\ Y\end{pmatrix} = \underbrace{\begin{bmatrix}0 & 0 & ig\\ 0 & -\Gamma & -ig\\ 2ig & -2ig & -\Gamma/2\end{bmatrix}}_{\hat{L}}\begin{pmatrix}\rho_{22}\\ \rho_{11}\\ Y\end{pmatrix}$$

Solution to this equation

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} e^{\lambda t} \Rightarrow \quad \lambda \vec{\xi} = \hat{L} \vec{\xi}$$

$$\det[\lambda \hat{I} - \hat{L}] = 0.$$

$$\det \begin{bmatrix} \lambda & 0 & -ig \\ 0 & \lambda+\Gamma & ig \\ +2ig & 2ig & \lambda+\Gamma/2 \end{bmatrix} = (\lambda + \Gamma/2)(\lambda^2 + \lambda\Gamma + 4g^2) = 0$$

$$\lambda_0 = -\Gamma/2 \qquad \lambda_{1,2} = -\Gamma/2 \pm \sqrt{\Gamma^2/4 - 4|g|^2}$$

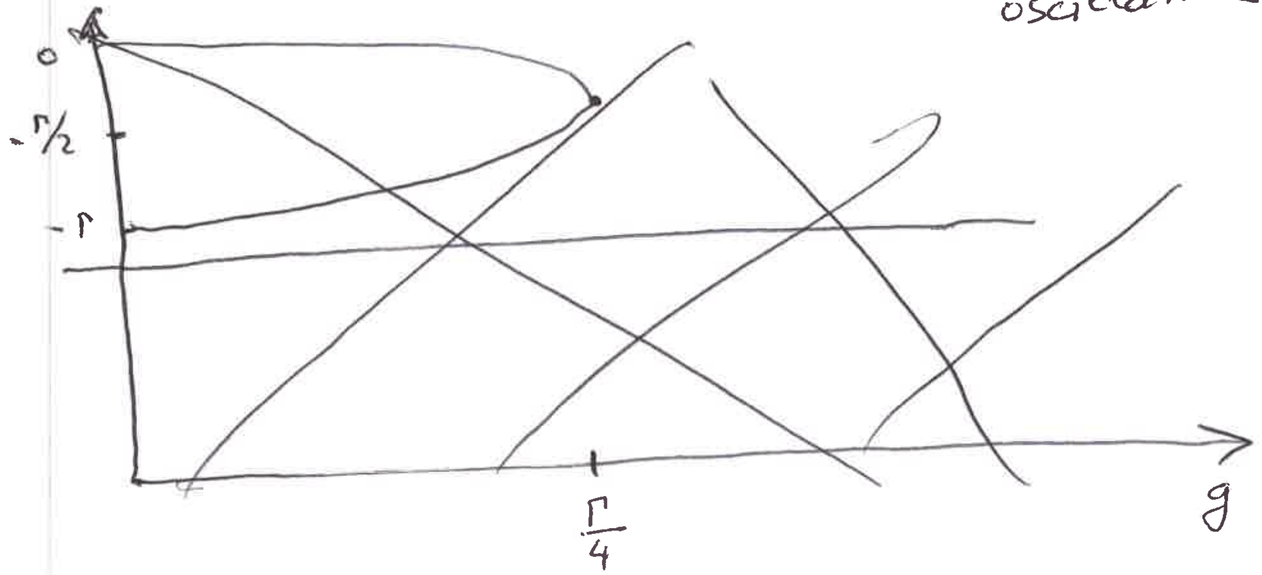
Solution $\quad$ & $\mathcal{D} = \sqrt{\Gamma^2 -}$

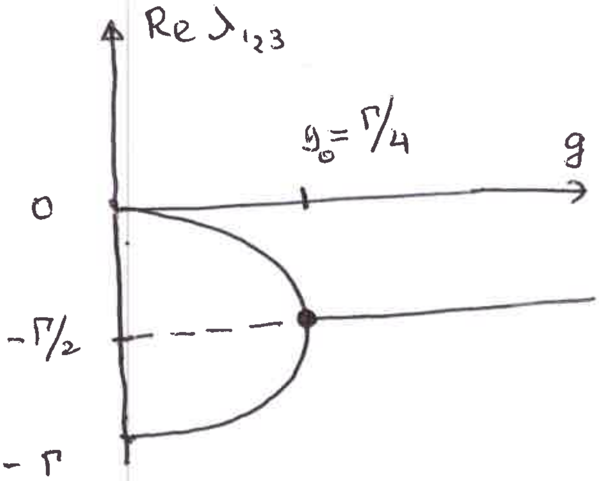
$$\begin{pmatrix} \rho_{11} \\ \rho_{22} \\ Y \end{pmatrix} = \vec{\xi}_0 \Big\{ e^{-\Gamma/2 t} + \vec{\xi}_1 e^{\lambda_1 t} + \vec{\xi}_2 e^{\lambda_2 t}$$

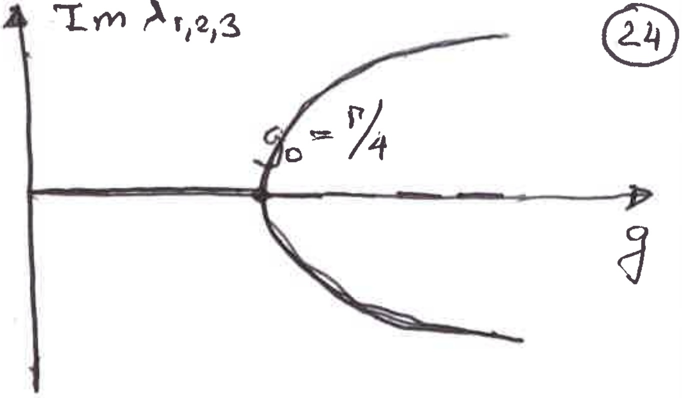
Regime of strong/weak coupling

$\Gamma^2/4 > 4|g|^2$ $(\Gamma > 4|g|)$ → weak coupling $\lambda_{1,2}$ - real, no oscillations

$\Gamma^2/4 < 4|g|^2$ $(\Gamma < 4|g|)$ → strong coupling $\lambda_{1,2}$ - complex, oscillations




Re λ1,2,3
g0 = Γ/4
g
0
-Γ/2
-Γ


Im λ1,2,3
g0 = Γ/4
g

Phenomenological execution of the kernel

- Two-level system $|1\rangle$ and $|2\rangle$

$$H = \hbar g(|1\rangle\langle 2| + |2\rangle\langle 1|)$$

$$|\psi\rangle = A|1\rangle + B|2\rangle$$

$$\dot{A} = -\int_0^t K(t-t')\,A(t')\,dt' \qquad K(t) = 4|g|^2$$

$$\dot{A} = 4|g|^2 \int_0^t e^{-\Gamma/2\,(t-t')}\,A(t')\,dt'$$

$$\Rightarrow \boxed{sA(s) - A(0) = -\frac{4|g|^2}{\Gamma/2 + s}\,A(s)}$$

$$A(s) = \frac{A(0)}{s + \frac{4|g|^2}{\Gamma/2+s}} \Rightarrow e^{-\Gamma/2 \pm \sqrt{\Gamma^2/4 + 4|g|^2}\; t}$$

time dependence is the same

Conclusion → decay is related to the memory depth, longer memory → coherent oscillations
shorter memory → incoherent decay.

Феноменологический подход Линблада.

Нам нужно найти самый простой способ описать ~~потери~~ релаксации в системе. То есть надо решить уравнение для приведенной матрицы

$$\dot{\hat\rho}_s = -\frac{i}{\hbar}[\hat H, \hat\rho_s] + \hat D[\hat\rho_s]$$

Но как быть, если мы практически ничего не знаем о ванне (термованне, окружающей среды)? Можно принять самое простое приближение: приближение скорости ~~потерь~~ релаксации

<u>Простой пример:</u> уравнение осциллятора

$$\dot f = -i\omega f \underset{\text{решение}}{\Longrightarrow} f = e^{-i\omega t} f_0$$

Можем добавить скорость потерь γ

$$\dot f = -i\omega f \boxed{-\gamma f} \Longrightarrow f = e^{-i\Omega t} f_0$$

$$\Omega = \omega - i\gamma$$

Как сделать такое для уравнения Лиувилля - фон Неймана?

$$\dot\rho_s = -\frac{i}{\hbar}[H, \rho_s] + \hat\gamma\hat\rho_s$$, где $\hat\gamma$ - матрица

Можно ли так делать? В принципе, да, но(!) мы должны удовлетворить ряду условий

Главные условия:

1. $\mathrm{Tr}[\hat{\rho}_s] = 1$ в любой момент времени (нормировк

2. $\rho_{s\,ii}\,\rho_{s\,jj} \geq |\rho_{s\,ij}|^2$ в любой момент времени (состояние квантовое) не может быть < ограничение сверху

Как выбрать $\hat{\gamma}$? Линдблад предложил следующи способ: Если выбрать $\hat{D}$ в виде

$$\hat{D}[\hat{\rho}_s] = \sum_{n,m} \frac{\gamma_{mn}}{2} \left[2\hat{L}_n \hat{\rho}_s \hat{L}_m^+ - \hat{\rho}_s \hat{L}_m^+ \hat{L}_n - \hat{L}_m^+ \hat{L}_n \rho_s \right]$$

то условия 1, 2 удовлетворяются. Здесь γ_{nm} – константы, $\hat{L}_n$ – операторы, действующие в пространств состояний системы. Это выражение можно записать в диагональном виде, если выбрать базис операторов $L_n \rightarrow L_j$

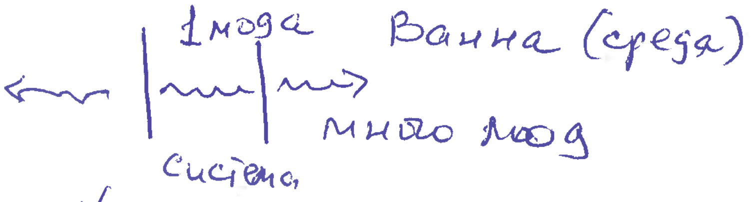
$$\hat{D}[\hat{\rho}_s] = \sum_j \frac{\gamma_j}{2} \left(2\hat{L}_j \hat{\rho}_s \hat{L}_j^+ - \hat{\rho}_s \hat{L}_j^+ \hat{L}_j - \hat{L}_j^+ \hat{L}_j \hat{\rho}_s \right)$$

Тогда $\hat{L}_j$ имеет смысл операторов процессов ~~которы~~ релаксаци

а γ_j – соответствующая константа –

– скорость ~~которь~~ релаксации

Вспомним результат вычисления для потери из резонатора согласно теории возмущений?



$$\hat H_I = \hbar \sum_k e^{-i(\omega_k-\omega)t}\, b_k a^+ + \hbar \sum_k g_k^* e^{i(\omega_k-\omega)t}\, b_k^+ a$$

Мы получили в том расчете

$$\hat D[\hat\rho_s] = \frac{\gamma}{2}\cdot(n_\omega+1)\left(2\hat a\hat\rho_s\hat a^+ - \hat\rho_s\hat a^+\hat a - \hat a^+\hat a\hat\rho_s\right)$$

$$+ \frac{\gamma}{2} n_\omega\left(2\hat a^+\hat\rho_s\hat a - \hat\rho_s\hat a\hat a^+ - \hat a\hat a^+\hat\rho_s\right)$$

Сравнимая где $n_\omega = \dfrac{1}{e^{\hbar\omega/T}-1}$ - число фотонов в ванне

Сравнивая с феноменологической формулой Линблада получаем

$\hat L_1 = \hat a$ — физический смысл: фотон улетает из системы

$\hat L_2 = a^+$ — физический смысл: фотон прилетает в систему

[illegible]

Заметим:

1. Константа γ_j зависит от

а) состояние окружающей среды – n_ω.
(зависит от температуры)

б) от ~~элементы~~ вероятности перехода

$$\gamma = 2\pi \sum_k{}' |g_k|^2 \delta(\omega - \omega_k)$$ – вероятность перехода в единицу времени, расчитанную по "Золотому правилу Ферми" (теория возмущений до второго порядка)

Важный частный пример: в ~~волне~~ среде нет фотонов (температура = 0)

$$n_\omega = \frac{1}{e^{\frac{\omega\hbar}{T}}-1} \xrightarrow[T\to 0]{} 0.$$

Тогда фотон может только уйти из системы ($n_\omega+1 = 1$), и тогда остается только один процесс $\hat{L}_1 = \hat{a}$.

Заметим: ~~[illegible]~~ модель Линдблада означает полное пренебрежение памятью ~~системы~~ ~~[illegible]~~ {термостата}: если фотон выскочил из системы в термостат, то он тут же "забыл, откуда он"

То есть, фотон, вылетев из системы, многораз столкнулся ~~из~~ с другими компонентами (частицами, стенками, возбуждениями) и "термализовался" — то есть его состояние описывается температурным распределением Гиббса

$$\hat{\rho}_{ph}(k) = \frac{1}{Z_k} e^{-\frac{\hat{H}_k}{T}} = \frac{1}{Z_k} \sum_{j_k=0}^{+\infty} e^{-\frac{\hbar\omega_k j_k}{T}} |j_k\rangle\langle j_k|$$

k – мода фотона

причем, это распределение возникает "мгновенн сразу после вылета из системы (нет "памяти").

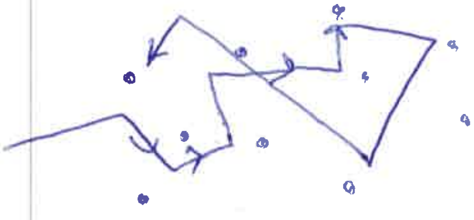
2 типа процессов ~~потери~~ релаксаци

До сих пор мы рассматривали процессы при которых фотон улетал из системы, ~~+~~ или прилетал в нее. Таким образом, менялась энергия системы (± ħω). Такие процессы наиболее легко представить для описания ~~потерь~~ релаксации. Однако, существует еще один тип процессов ~~потерь~~ релаксации квантовой когерентности, который не связан с ~~потерей~~ изменением энергии.

Что это за процессы?

Это процессы, приводящие к ~~потере~~ релаксации фазы квантового состояния.

Простой пример: частица движется в случайном потенциале (случайно разбросанные точки



Частица не теряет энергию но полностью "забывает" направление первоначального движения.

Простой квантовый аналог: двухмерная систем

$$|\psi\rangle = \alpha|\psi_\alpha\rangle + \beta|\psi_\beta\rangle$$

ее матрица плотности дается простым выражением

$$\hat{\rho} = |\psi\rangle\langle\psi| = \cancel{\Sigma}\ \rho_{\alpha\alpha}|\psi_\alpha\rangle\langle\psi_\alpha| + \rho_{\beta\beta}|\psi_\beta\rangle\langle\psi_\beta|$$

$$+ \rho_{\alpha\beta}|\psi_\alpha\rangle\langle\psi_\beta| + \rho_{\beta\alpha}|\psi_\beta\rangle\langle\psi_\alpha|$$

или, в матричной форме (в представлении базиса $|\psi_\alpha\rangle, |\psi_\beta\rangle$)

$$\hat{\rho} = \begin{bmatrix} \rho_{\alpha\alpha} & \rho_{\alpha\beta} \\ \rho_{\beta\alpha} & \rho_{\beta\beta} \end{bmatrix}$$

где $\rho_{\alpha\alpha} = \alpha\alpha^* = |\alpha|^2$

$\rho_{\beta\beta} = \beta\beta^* = |\beta|^2$

$\rho_{\alpha\beta} = \alpha\beta^*$ $\rho_{\beta\alpha} = \beta\alpha^* = \rho_{\alpha\beta}^*$

Выразим коэффициенты через амплитуду и фазу

$$\alpha = |\alpha| e^{i\varphi_\alpha}; \quad \beta = |\beta| e^{i\varphi_\beta}$$

тогда

$$\hat{\rho} = \begin{bmatrix} |\alpha|^2 & |\alpha||\beta| e^{i(\varphi_\alpha-\varphi_\beta)} \\ |\alpha||\beta| e^{i(\varphi_\beta-\varphi_\alpha)} & |\beta|^2 \end{bmatrix}$$

Допустим теперь, что $\varphi_\alpha - \varphi_\beta$ - случайным образом меняется (быстро). Усреднив за любое время (характерное время измерений, медленное) получим

$$\langle \rho_{\alpha\alpha} \rangle = \frac{1}{T}\int_0^T \rho_{\alpha\alpha}(t)\,dt = |\alpha|^2$$

$$\langle \rho_{\beta\beta} \rangle = \frac{1}{T}\int_0^T \rho_{\beta\beta}(t)\,dt = |\beta|^2$$

$$\langle \rho_{\alpha\beta} \rangle = \frac{1}{T}\int_0^T e^{i(\varphi_\alpha-\varphi_\beta)}|\alpha||\beta|\,dt \approx |\alpha||\beta|\int_0^{2\pi} e^{i\varphi}\,d\varphi = 0$$

Тогда наша матрица, которой эффективно описывает система

$$\langle \hat{\rho} \rangle \Rightarrow \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}$$

Состояние из чистого становится "грязным" (смешанным). А энергия (средняя)

$$\langle E\rangle = Tr[\hat H \hat\rho] = Tr[\hat H(|\psi_\alpha\rangle\langle\psi_\alpha|\rho_{\alpha\alpha} + |\psi_\beta\rangle\langle\psi_\beta|\rho_{\beta\beta}$$

$$+|\psi_\alpha\rangle\langle\psi_\beta|\rho_{\alpha\beta} + |\psi_\beta\rangle\langle\psi_\alpha|\rho_{\beta\alpha})]$$

$\hat H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle$ $\hat H|\psi_\beta\rangle = E_\beta|\psi_\beta\rangle$ (так выбрал базис)

$$\Rightarrow \langle E\rangle = Tr[E_\alpha|\psi_\alpha\rangle\langle\psi_\alpha|\rho_{\alpha\alpha} + E_\beta|\psi_\beta\rangle\langle\psi_\beta|\cdot\rho_{\beta\beta} +$$

$$+E_\alpha|\psi_\alpha\rangle\langle\psi_\beta|\rho_{\alpha\beta} + E_\beta|\psi_\beta\rangle\langle\psi_\alpha|\rho_{\beta\alpha}] =$$

* Замечание $Tr[\hat f] = \cancel{\sum_j} \langle\psi_\alpha|\hat f|\psi_\alpha\rangle + \langle\psi_\beta|\hat f|\psi_\beta\rangle$

$$\Rightarrow \langle E\rangle = E_\alpha\rho_{\alpha\alpha} + E_\beta\rho_{\beta\beta}$$

это выражение не зависит от $\rho_{\alpha\beta}$ $(\rho_{\beta\alpha})$ и таким образом энергия для состояний, описываемых матрицами

$$\hat\rho_1 = \begin{bmatrix}\rho_{\alpha\alpha} & \rho_{\alpha\beta}\\ \rho_{\beta\alpha} & \rho_{\beta\beta}\end{bmatrix} \quad и \quad \hat\rho_2 = \begin{bmatrix}\rho_{\alpha\alpha} & 0\\ 0 & \rho_{\beta\beta}\end{bmatrix}$$ — одинаков

Но разница в том, что $\hat\rho_1$ описывает чистое состояние замкнутой (изолированной системы)

а $\hat\rho_2$ — состояние открытой системы с ~~потере~~ релаксаци

Только ~~потери~~ релаксация эта — квантовая дефазировка ("чистая" дефазировка)

Такие процессы можно описать формализмом Линблада, если взять $\hat{L}_j$ (оператор релаксации ~~потери~~) который описывает процесс без изменения ~~потери~~ энерги

Пример: процессы $\hat{L}_1 = \hat{a}$ и $\hat{L}_2 = \hat{a}^+$ —

~~хро~~ процессы с изменением энергии

$$\hat{a} \Rightarrow |j\rangle \longrightarrow |j-1\rangle \quad (\hat{a}|j\rangle = \sqrt{j}\,|j-1\rangle)$$

$$\hat{a} \Rightarrow |j\rangle \rightarrow |j+1\rangle \quad (\hat{a}^+|j\rangle = \sqrt{j+1}\,|j+1\rangle)$$

Процессы без изменения энергии описываютс проекционными операторами

$$\hat{P}_j = |j\rangle\langle j| \qquad \hat{P}_j^+ = \hat{P}_j \qquad \hat{P}_j^2 = \hat{P}_j$$

Таким образом, чистая дефазировка описыва

$$D_j[\hat{\rho}_s] = \frac{\gamma_j}{2}\left(2\hat{P}_j\hat{\rho}_s\hat{P}_j^+ - \hat{\rho}_s\hat{P}_j^+\hat{P}_j - \hat{P}_j^+\hat{P}_j\hat{\rho}_s\right) =$$

$$= \frac{\gamma_j}{2}\left(2P_j\hat{\rho}_s P_j - \hat{\rho}_s\hat{P}_j - \hat{P}_j\hat{\rho}_s\right)$$

Эти операторы тоже удовлетворяют всем соотношениям, необходимым для сохранения структуры матрицы плотности

Решение уравнений для $\hat{\rho}_s$ с операторами Линбладе

Замечание: уравнение $\dot{\rho}_s = -\frac{i}{\hbar}[\hat{H},\hat{\rho}_s] + \hat{D}[\hat{\rho}_s]$

– матричное уравнение

Общий метод решения: выбрать базис состояни системы и "спроецировать" уравнение на этот базис ("представление" уравнения)

Пример 1: рассмотрим 1 фотон в резонаторе который может выскочить с вероятностью (в единицу времени) γ.

Состояния (базис): $|0\rangle$ – нет фотонов
$|1\rangle$ – 1 фотон

Матрица плотности $\rho = \rho_{00}|0\rangle\langle 0| + \rho_{11}|1\rangle\langle 1| +$
$+ \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0|$

Гамильтониан $\hat{H}_F = \hbar\omega\hat{a}^+\hat{a} = \hbar\omega|1\rangle\langle 1| + 0$

Допустим, что мы рассматриваем процесс с потерей фотона только $\hat{L} = \cancel{\gamma}\hat{a}$

Матричная форма в этом представлении

$$\hat\rho = \begin{bmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{bmatrix} \qquad \hat H = \hbar\omega\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \qquad \hat a = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

$$\hat a^+ = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

Вычислим $\hat D[\hat\rho] = \frac{\gamma}{2}\Big[2\hat a\hat\rho\hat a^+ - \hat\rho\hat a^+\hat a - \hat a^+\hat a\,\hat\rho\Big] =$

$$= \frac{\gamma}{2}\left[2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - \right.$$

$$\left. - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\right] = \frac{\gamma}{2}\begin{bmatrix}2\rho_{11} & -\rho_{01}\\ -\rho_{10} & -2\rho_{11}\end{bmatrix}$$

$$[\hat H, \hat\rho] = \begin{pmatrix}0 & 0\\ 0 & \hbar\omega\end{pmatrix}\begin{pmatrix}\rho_{00} & \rho_{10}\\ \rho_{01} & \rho_{11}\end{pmatrix} - \begin{pmatrix}\rho_{00} & \rho_{10}\\ \rho_{01} & \rho_{11}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \hbar\omega\end{pmatrix} = \hbar\omega\begin{pmatrix}0 & -\rho_{01}\\ \rho_{10} & 0\end{pmatrix}$$

$$\Rightarrow \begin{bmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{bmatrix} = +i\omega\begin{bmatrix}0 & \rho_{01}\\ -\rho_{10} & 0\end{bmatrix} + \gamma\begin{bmatrix}\rho_{11} & -\rho_{01}/2\\ -\rho_{01}/2 & -\rho_{11}\end{bmatrix}$$

$$\Rightarrow \begin{cases}\dot\rho_{00} = \gamma\rho_{11}\\ \dot\rho_{11} = -\gamma\rho_{11}\end{cases} \qquad \begin{array}{l}\dot\rho_{01} = i\omega\rho_{01} - \gamma\rho_{01}\\ \dot\rho_{10} = -i\omega\rho_{10} - \gamma\rho_{10}\end{array}$$

комплексно сопряжены уравнения

$$\Downarrow$$

$$\rho_{11}(t) = e^{-\gamma t}\rho_{11}(0)$$

$$\rho_{00}(t) = \rho_{00}(0) + \frac{1 - e^{-\gamma t}}{\ }\rho_{11}(0)$$

$$\rho_{11}(0) + \rho_{00}(0) = 1$$

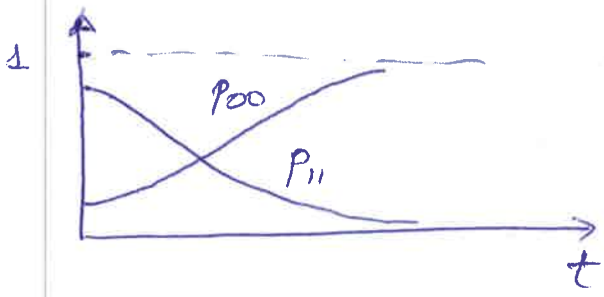
$$\Rightarrow \rho_{00}(0) = f$$

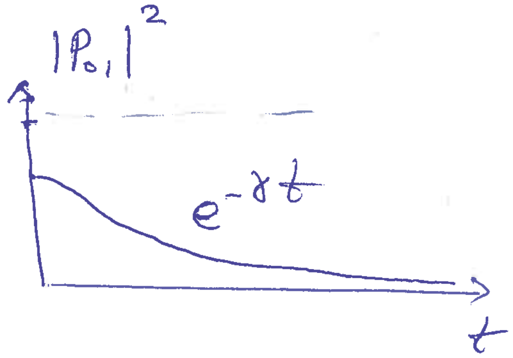
$$\rho_{11}(0) = 1 - f$$

$$\Rightarrow f + (1 - e^{-\gamma t})(1 - f) = 1 - (1 - f)e^{-\gamma t}$$

$$\rho_{01} = e^{i\omega t - \gamma/2\, t}\rho_{01}$$

График временной зависимости





Пример 2: 1 фотон в резонаторе, выскочить не может, но нарушается фаза

Релаксационный процесс — чистая дефазировка

Оператор Линдблада $\hat{L} = |1\rangle\langle 1| \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Тогда $\hat{D}[\hat{\rho}] = \frac{\gamma}{2}\left[2\hat{L}\hat{\rho}\hat{L} - \hat{L}\hat{\rho} - \hat{\rho}\hat{L}\right] =$

$$= \frac{\gamma}{2}\left[2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} - \right.$$

$$\left. - \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \frac{\gamma}{2}\begin{pmatrix} 0 & -\rho_{01} \\ -\rho_{10} & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{\rho}_{00} & \dot{\rho}_{01} \\ \dot{\rho}_{10} & \dot{\rho}_{11} \end{pmatrix} = i\omega\begin{pmatrix} 0 & \rho_{01} \\ -\rho_{10} & 0 \end{pmatrix} - \frac{\gamma}{2}\begin{pmatrix} 0 & \rho_{01} \\ \rho_{10} & 0 \end{pmatrix}$$

$$\begin{cases} \dot{\rho}_{00} = 0 \\ \dot{\rho}_{11} = 0 \end{cases} \qquad \begin{matrix} \dot{\rho}_{01} = i\omega\rho_{01} - \frac{\gamma}{2}\rho_{01} \\ \dot{\rho}_{10} = -i\omega\rho_{10} - \frac{\gamma}{2}\rho_{10} \end{matrix}$$

Surface plasmons

metall | dielectric

Modes

Maxwel equation

$$\begin{cases} \text{rot}\, E = -\frac{1}{c}\frac{\partial B}{\partial t} \\ \text{rot}\, B = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial E}{\partial t} \\ \cancel{\text{rot}}\, B = 0. \\ \text{div}\, E = 4\pi \rho. \end{cases}$$

$\vec{j}$ and ρ are current and charge density in the media

$$\begin{cases} \vec{j} = e \int \vec{v}\,(n - n_0)\, d^3p. \\ \rho = e \int (n - n_0)\, d^3p. \end{cases}$$

~~here~~ $n(x, p, t)$ - distribution function of charge particles in the media

$n_0(x, p, t) \equiv n_0(p)$ - equilibrium function in the media

Non-equilibrium due to external fields

If the system is "almost" classical

n is defined by the Boltzmann ~~distribution~~ equation

$$\underbrace{\frac{dn}{dt}} = I[n] \leftarrow$$ ~~collision term~~ relaxation term (collisions)

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial p}\dot{\vec{p}} + \frac{\partial n}{\partial x}\dot{\vec{x}} = I[n]$$

$$\downarrow$$

$$\dot{\vec{p}} = e\vec{E} + \cancel{\frac{e}{c}[\vec{v}\times\vec{B}]}$$

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \vec{p}}\cdot e\vec{E} + \frac{\partial n}{\partial x}\cdot\vec{v} = I[n]$$

~~look for the solution [illegible] = n − n₀ = f~~

~~$\frac{\partial f}{\partial t} + \frac{\partial n_0}{\partial p} eE$ [illegible]~~ $I[n] \approx \frac{n-n_0}{\tau}$ = simple approximat

Linear approximation

$f = n - n_0$

$$\boxed{\frac{\partial f}{\partial t} + \frac{\partial n_0}{\partial p}\cdot\vec{E}e + \frac{\partial f}{\partial x}\vec{v} = \frac{f}{\tau}}$$

=> ~~[illegible]~~ ~~[illegible]~~ ~~f(x,p) ⟷ f(k,p) [illegible]~~

Fourier transform $f(x,p,t) \longrightarrow f(x,p,\omega)$

$e^{ikx - i\omega t}$

$$f = -i\,\frac{e\vec{E}}{\omega - k v + i/\tau}\,\frac{\partial n_0}{\partial \vec{p}}$$ ③

$\Rightarrow$ ρ and j as a linear function of $\vec{E}$

$$\begin{cases} \rho(k,\omega) = -ie^2 \displaystyle\int \frac{\vec{E}}{\omega - kv + i/\tau}\,\frac{\partial n_0}{\partial \vec{p}}\, d\vec{p}^{\,3} \\ \vec{j}(k,\omega) = -ie^2 \displaystyle\int \frac{\vec{v}_z \left(\vec{E}\cdot\frac{\partial n_0}{\partial \vec{p}}\right)}{\omega - kv + i/\tau}\, d^3p. \end{cases}$$

$$\cancel{E} \quad \vec{D} = \vec{E} - 4\pi P$$

$$\boxed{\operatorname{div} P = \rho}$$

$\Rightarrow$ $\vec{D} = \hat{\varepsilon}\,\vec{E}$ $\hat{\varepsilon}(k,\omega)$ – dielectric permittivi tensor.

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + \frac{4\pi i}{\omega}\left\{ -ie^2 \int \frac{v_\alpha}{\omega - \vec{k}\vec{v}}\,\frac{\partial n_0}{\partial p_\beta}\, d^3p \right\}$$

$\Rightarrow$ In the linear approximation we can substitute the reaction of the medium by known ~~functions~~ dielectr responce function.

$$\begin{cases} \mathrm{rot}\, E = -\frac{1}{c}\frac{\partial B}{\partial t} \\ \mathrm{rot}\, B = \frac{1}{c}\frac{\partial D}{\partial t} \\ \mathrm{div}\, B = 0. \\ \mathrm{div}\, D = 0. \end{cases}$$

magnetic response is neglected, external charges are absent.

~~For the dielectric mater.~~

$$\vec{D} = \check{\varepsilon} * \vec{E}$$

For dielectric material $\check{\mathcal{E}} = \varepsilon_d \check{I}$

For metal $\check{\varepsilon} = \check{I}\left(1 - \frac{\Omega_p^2}{\omega^2}\right).$

$\Omega_p^2 = \frac{4\pi n_e e^2}{m}$ — plasma frequency

Solution is exact

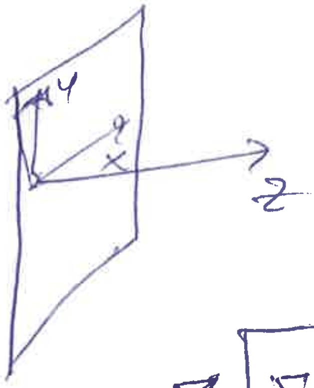
$$E_x^{(L,R)} = \sigma \frac{q^2 - \varepsilon_{(L,R)}\omega^2/c^2}{\varepsilon q} G$$

~~$e^{i\vec{q}\vec{r} + ik_z|z|}$~~

$$E_z^{(L,R)} = \sigma \frac{k_z}{\varepsilon_{(L,R)}} G$$

$$G^{(L,R)} = \frac{4\pi i}{k^2 - \varepsilon_{(L,R)}\omega^2/c^2}$$

$$\vec{k} = (q_x, q_y, k_z)$$



$$\boxed{\nabla A - \hat{\varepsilon} * \dot{\varphi} = 0.}$$

Lorenz!

$$E^{(L,R)}(\vec{r}) = \oint e^{iq_x x + iq_y y - i\omega t} \cdot \int dk_z \, E^{(L,R)}(k,\omega) \qquad ⑤$$

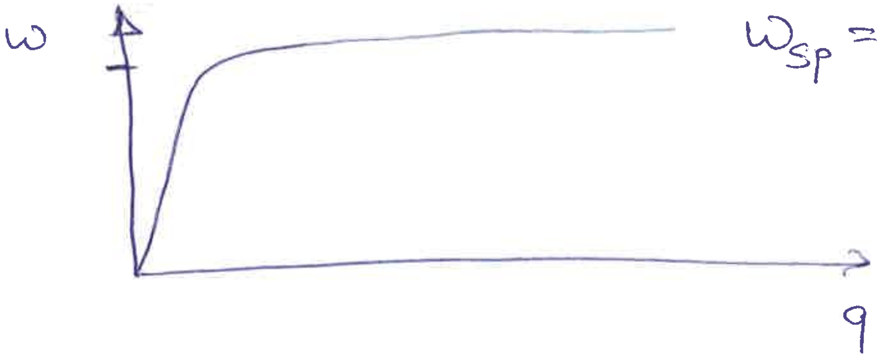
$$\Rightarrow \quad \begin{cases} E^{(L)}_{\cancel{z}x,y}(z=0) = E^{(R)}_{\cancel{z}x,y}(z=0). \\ D^{(L)}_{z}(z=0) = D^{(R)}_{z}(z=0). \end{cases}$$

$$\Downarrow$$

Dispersion relation

$$\frac{1}{\varepsilon_L}\sqrt{q^2 - \varepsilon_L \omega^2/c^2} = -\frac{1}{\varepsilon_R}\sqrt{q^2 - \varepsilon_R \omega^2/c^2}$$

$$\Rightarrow \boxed{\frac{1}{\varepsilon_L} + \frac{1}{\varepsilon_R} = \frac{\omega^2}{c^2 q^2}} \qquad \longrightarrow$$



$$\omega_{sp} = \frac{\cancel{k}\,\Omega_p}{\sqrt{1+\varepsilon_d}}$$

Quantization

$$\hat{\vec{E}} = \sum_q \sqrt{\frac{\hbar\omega_q}{2}} \; \vec{\mathcal{E}}_q(\vec{r}) \, \hat{a}_q + c.h.$$

Normalization factor

Energy of the mode must be $\hbar\omega_q$

Energy density of electromagnetic wave ⑥
in media

$$W = \frac{1}{16\pi}\int\left(E\left[\frac{\partial}{\partial\omega}(\omega\varepsilon)E\right] + B^2\right) d^3r.$$

$$W_q = \frac{1}{16\pi}\int\left\{\left(\vec{E}_q^{\,2}\varepsilon + \vec{E}_q^{\,2}\omega\frac{\partial\varepsilon}{\partial\omega} + B_q^2\right) d^2r =\right.$$

$$= \hbar\omega_q.$$

$$\boxed{\frac{1}{16\pi}\int\left\{\vec{E}^2\left(\varepsilon+\omega\frac{\partial}{\partial\omega}\varepsilon\right)+\vec{B}^2\right\}dr = 1}$$

normalization condition

$$\Rightarrow \hat{H}_{SPP} = \hbar\sum_q \omega_q\left(\hat{a}_q^+\hat{a}_q+\frac{1}{2}\right).$$ — Quantizd Hamiltonian.

Important: $\check{\varepsilon}$ does not contain decay

with decay

$$\varepsilon = 1 - \frac{\Omega_p^2}{\omega(\omega+i\,{}^1/_T)} \Rightarrow \omega_q = \omega_q' + i\omega_q''$$

You can still the quantization

$$\hat{H}_{SPP} = \hbar \sum_q \omega'_q \hat{a}_q^+ \hat{a}_q$$

but the damping has to be introduced on the level of the dynamic equations or the interaction with the bath

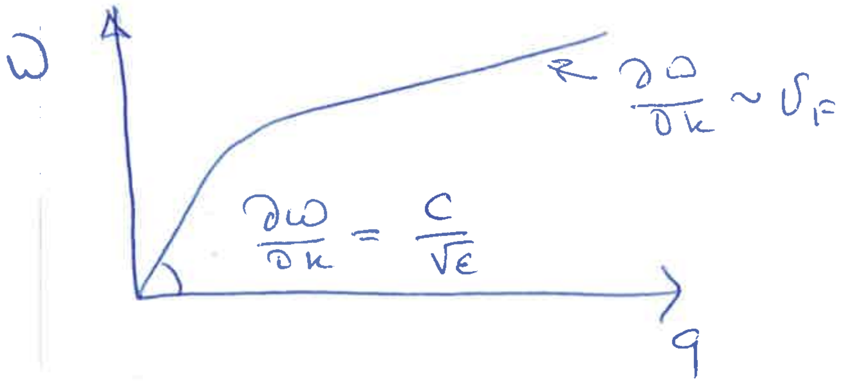
$$\Delta \hat{H}_I = \hbar \sum_q g_{qk} (a_q^+ b_k + a_q b_k^+).$$

$$\rightarrow \varepsilon_e = 1 + \frac{3\Omega_p^2}{k^2 v_F^2}\,\frac{\omega + i\Gamma}{\omega}\left\{1 - \frac{\omega + i\Gamma}{2 k v_F}\ln\left(\frac{\omega + i\Gamma + v_F k}{\omega + i\Gamma - v_F k}\right)\right\}$$

$$\varepsilon_{tr} = 1 - \frac{3\Omega_p^2}{2 k^2 v_F^2}\,\frac{\omega + i\Gamma}{\omega}\left\{1 - \frac{1}{2}\left(\frac{\omega + i\Gamma}{k v_F} - \frac{v_F k}{\omega + i\Gamma}\right)\ln\left(\frac{\omega + i\Gamma + v_F k}{\omega + i\Gamma - v_F k}\right)\right\}$$

$$\Omega_p^2 = \frac{4\pi n e^2}{m}$$

$$\varepsilon_{\alpha\beta} = \varepsilon^{tr}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) + \varepsilon^{l}\left(\frac{k_\alpha k_\beta}{k^2}\right).$$



Landau damping

$$\omega'' \sim -\Gamma \quad \boxed{v_F'' q}$$

– Linear dependence on q