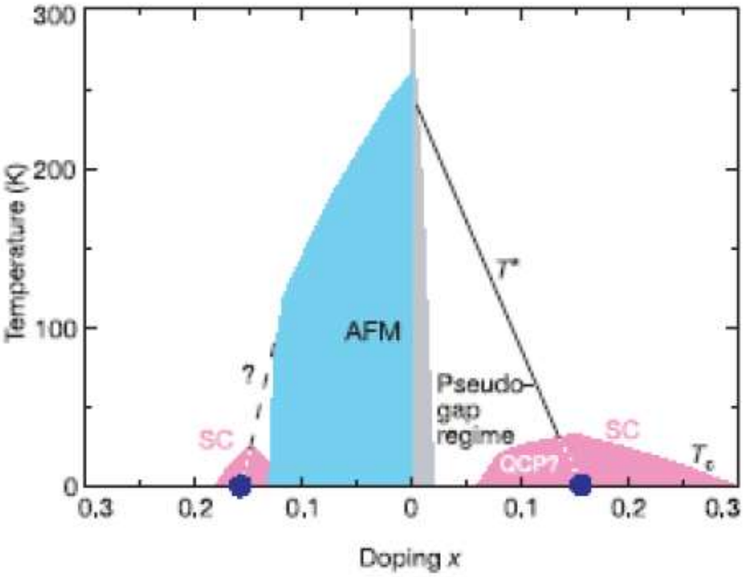
Introduction to the Physics of Correlated Electrons

Example: High Temperature Superconductors

L. Alff, *et. al.*, NATURE, VOL **422**, p.698 (2003)



Topics

- **metals with strong disorder**
- **weak localization**
- **anomalous magnetoresistance**
- **electron-electron interactions**
- **hopping coductivity**
- **Mott-Anderson metal-insulator transitions**
- **scaling theory of localization**
- **Kondo effect**

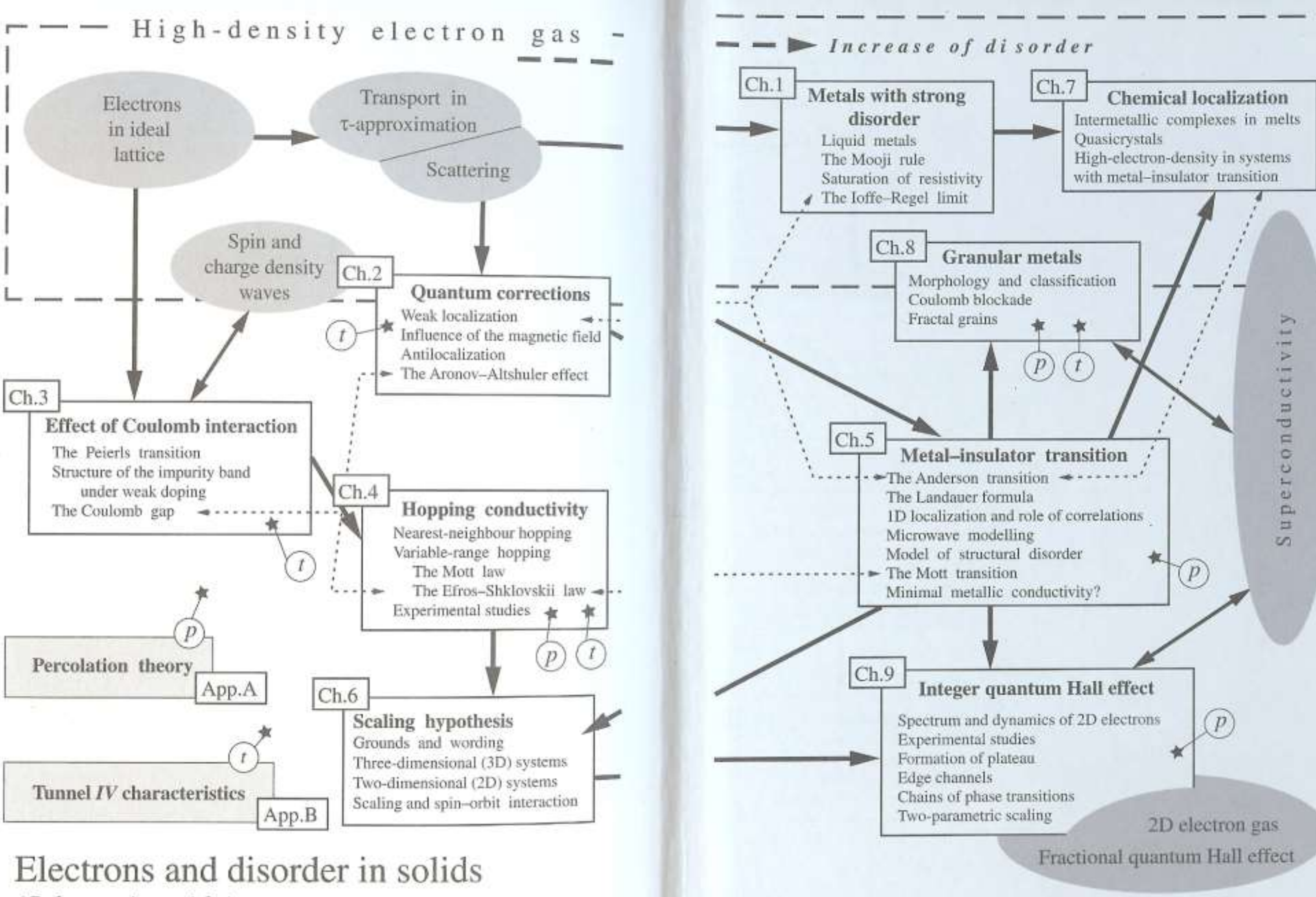
Textbook: V.F. Gantmakher
"Electrons and Disorder in Soilds",
Oxford University Press 2005
ISBN 0-19-856756-1

The structure of the course

- Lectures
- Homework assignments (3 sets)
- Presentations
- Oral exam

Final grade: assignments (1/4) + presentations (1/4) + oral exam (1/2)

Presentations should be based on recent research papers on a topic related to the course material.

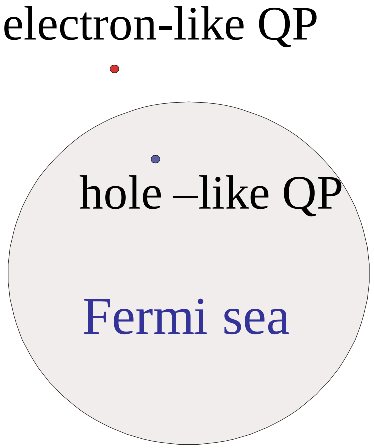


Electrons and disorder in solids

(Schematic guide)

Fermi liquid and quasiparticles

Quasiparticle concept (Landau 1956, 1957)



electron-like QP

hole –like QP

Fermi sea

Fermi liquid - a system of interacting Fermi particles

Quasiparticle (QP) – excitation in a Fermi liquid, it resembles an excitation in an ideal Fermi gas, but not equivalent

Due to interaction with other electrons and ions, quasiparticle **effective mass m^*** differs from the free electron mass m_e

In superconductors **effective charge e^*** also differs from electronic charge e

Excitation energy

$$\xi = p^2/2m^* - p_F^2/2m^* \approx v_F(p-p_F)$$

Quasiparticles have finite **lifetime** due to interaction with other electrons, phonons, etc.

Fermi Liquid

- *Fermi statistics*
- *Low temperatures*
- *Not too strong interactions*
- *Translation invariance*

Fermi Liquid

L.D. Landau, ”*Fermi-Liquid Theory*”
Zh. Exp. Teor. Fiz., v.30, p.1058 (1956)

- *Fermi statistics*
- *Low temperatures*
- *Not too strong interactions*
- *Translation invariance*

→ **Fermi Liquid**

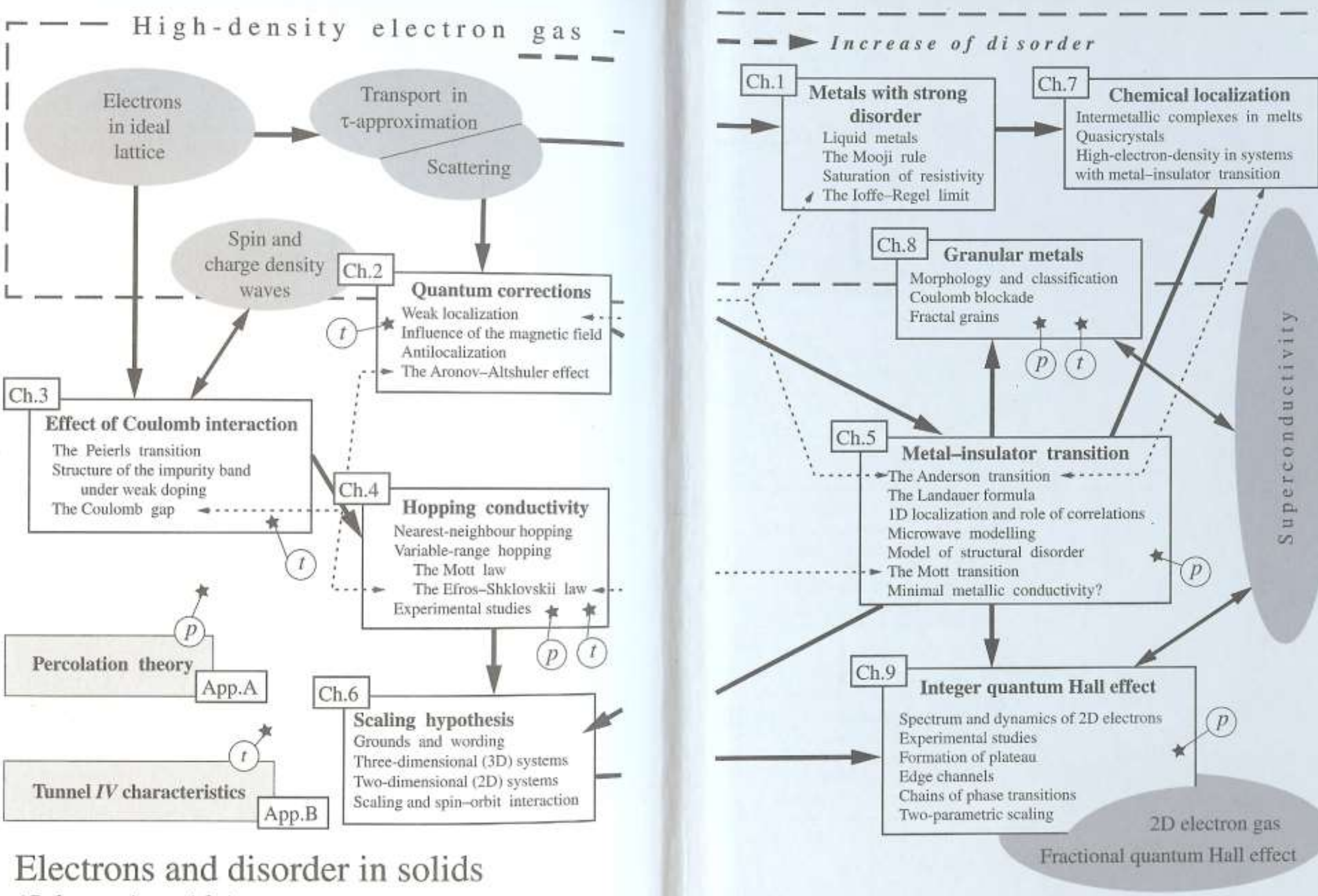
It means that

1. Excitations are **similar** to the excitations in a **Fermi-gas:**
   a) the **same** quantum numbers – momentum, **spin** ½ , **charge** e
   b) decay rate is **small** as compared with the excitation energy

2. Substantial renormalizations. For example, in a **Fermi gas**

$$\partial n/\partial\mu, \quad \gamma = c/T, \quad \chi/g\mu_B$$

are all equal to the one-particle density of states ν.
These quantities are **different** in a **Fermi liquid**

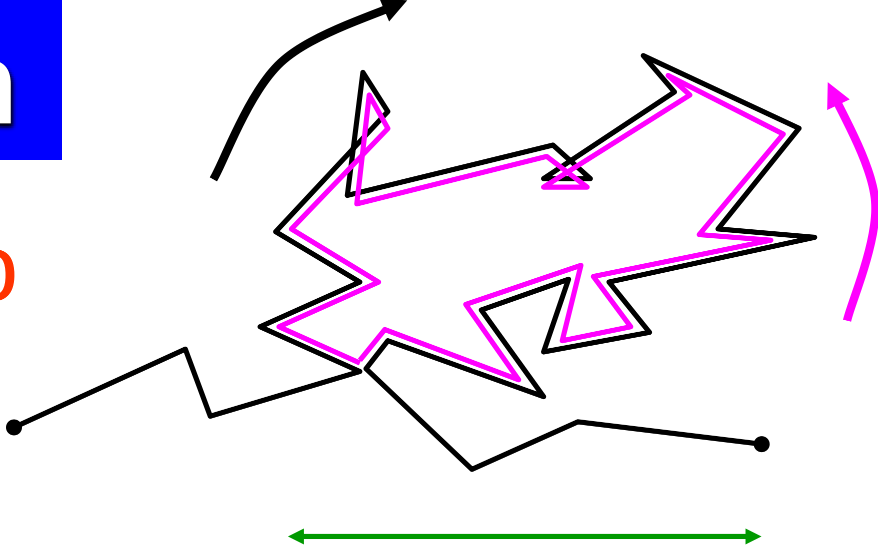


Electrons and disorder in solids

(Schematic guide)

Weak Localization

Quantum corrections to conductivity



$$\sigma = \frac{ne^2 l}{\hbar k_F}$$

Two types of electronic scattering

elastic scattering, probability **$1/\tau$**

inelastic scattering, probability **$1/\tau_\varphi$**

Phase φ of the wave function

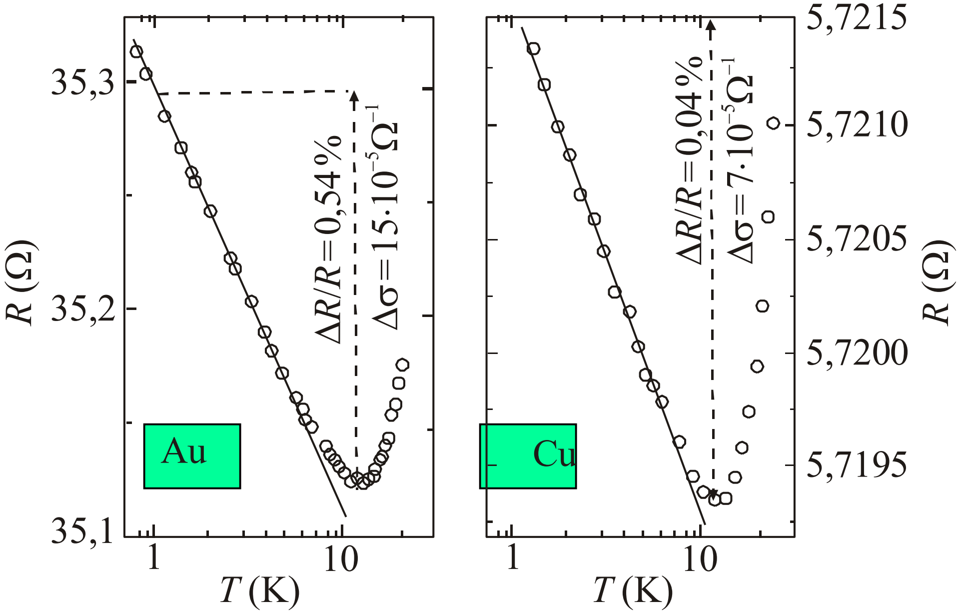
$$\psi \propto \exp(i\varepsilon t/\hbar)$$

$$\varphi = \varepsilon t/\hbar$$

$$\tau << \tau_\varphi$$

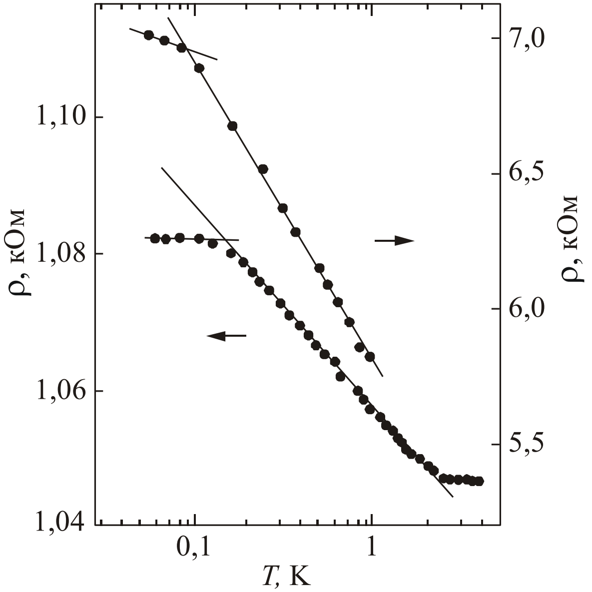
- phase coherence

Experiment



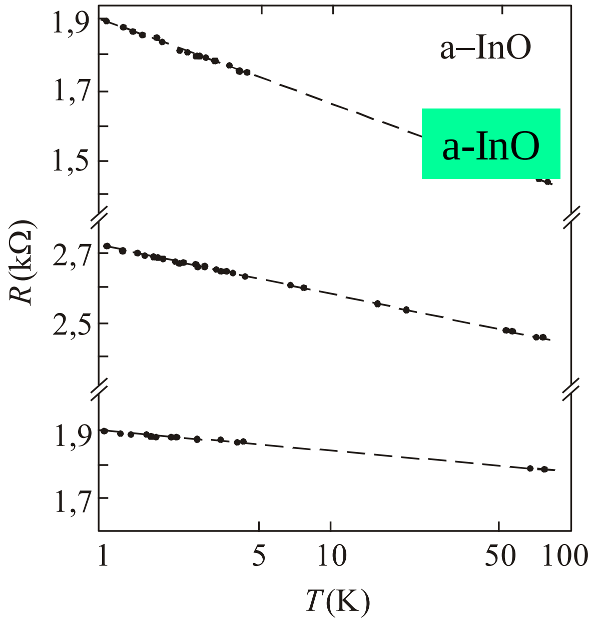
Cu – van der Dreis *et al.*, PRL **46,** 565 (1981)

Au – S. Dorozhkin, JETP Lett. **36,** 15 (1982)



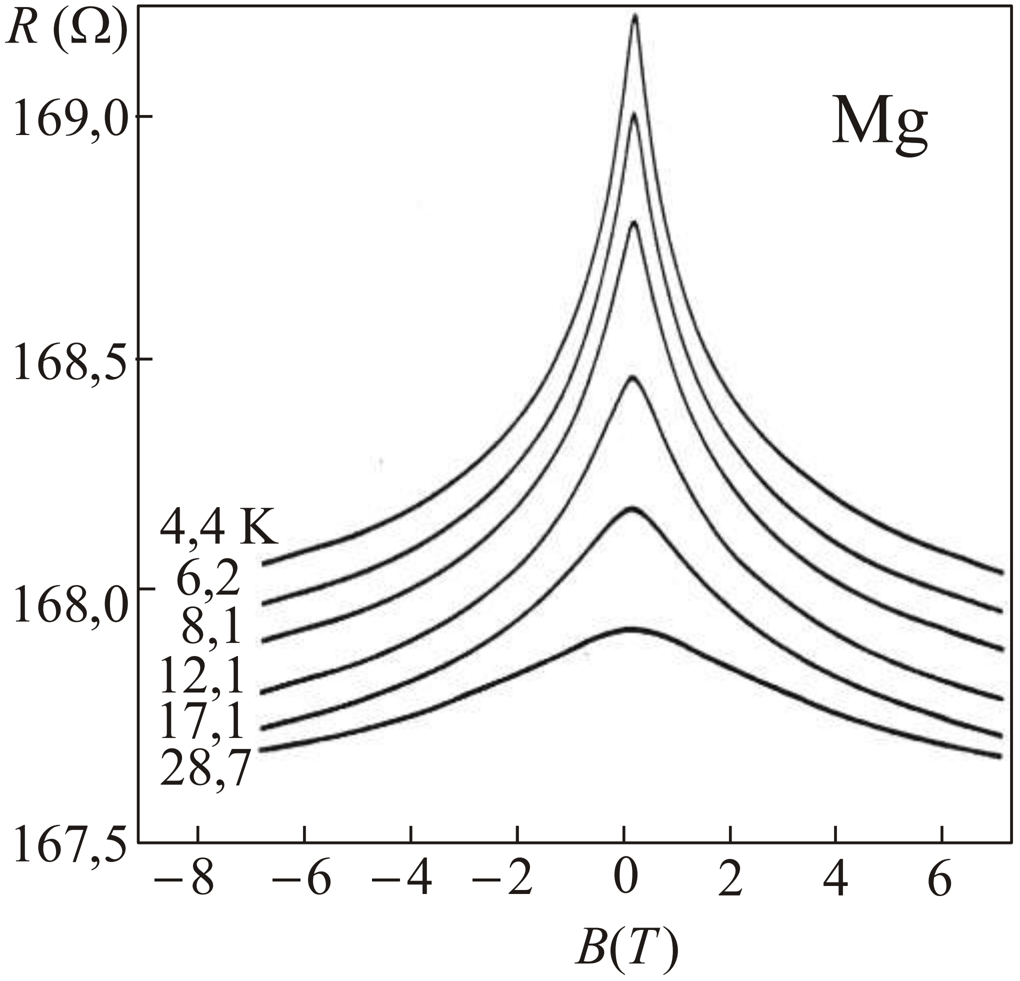
2DEG heterostructures

D.J.Bishop, D.C.Tsui, R.C.Dynes, PRL **44,** 1153 (1980)



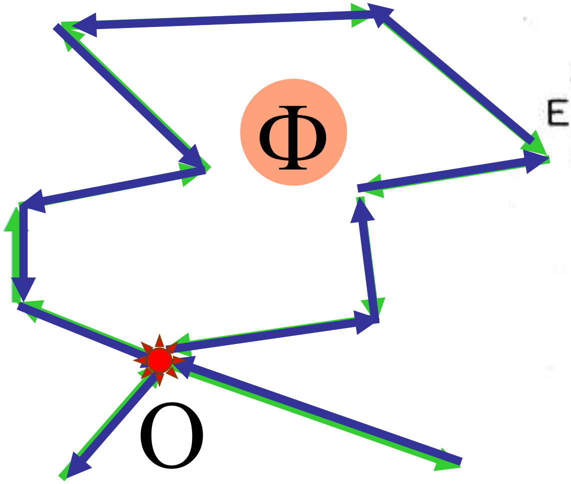
Z.Ovadyahu, Y.Imry
PRB 2**4,** 7440 (1981)

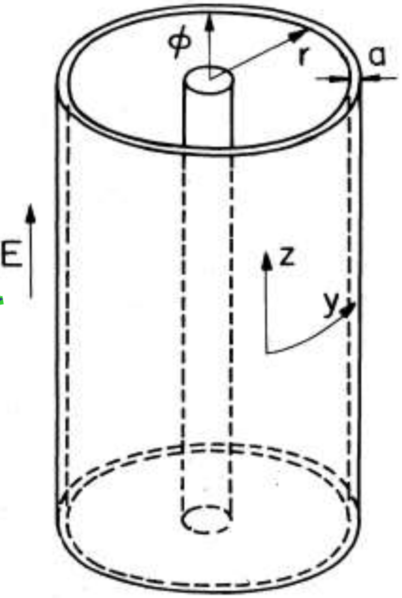
Anomalous (negative) magneto-resistance



G. Bergmann, Phys.Rep. **107,** 1 (1981)

Aharonov-Bohm effect





With magnetic field H

$$\varphi_1 - \varphi_2 = 2\pi\, \Phi/\Phi_0$$

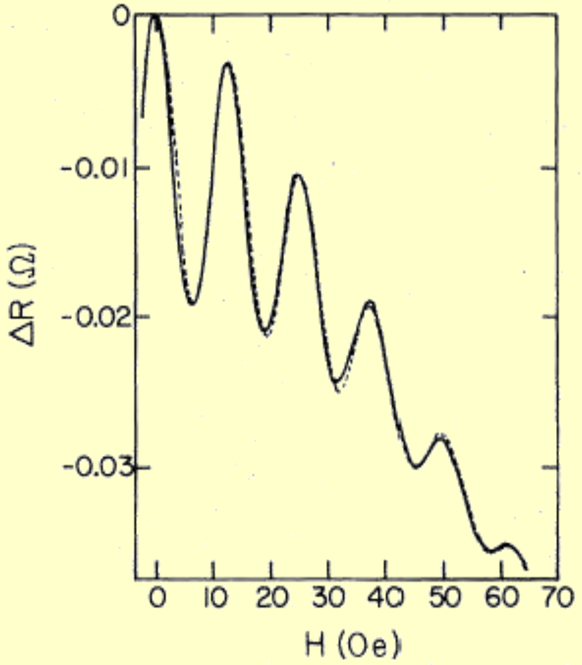
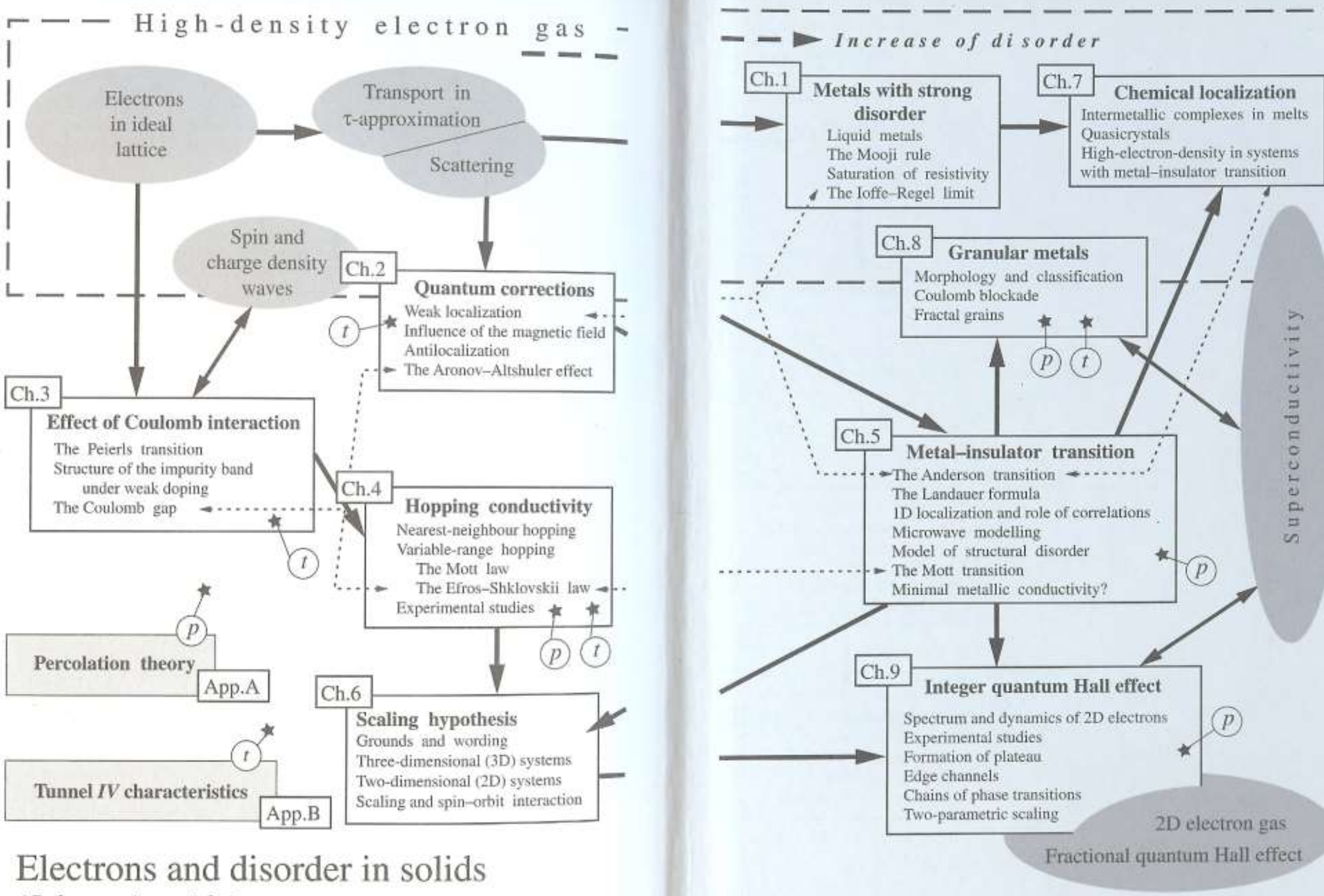


FIG. 8. Longitudinal magnetoresistance $\Delta R(H)$ at $T=1.1$ K for a cylindrical lithium film evaporated onto a 1-cm-long quartz filament. $R_{4.2}=2$ kΩ, $R_{300}/R_{4.2}=2.8$. Solid line: averaged from four experimental curves. Dashed line: calculated for $L_\varphi=2.2$ μm, $\tau_\varphi/\tau_{so}=0$, filament diameter $d=1.31$ μm, film thickness 127 nm. Filament diameter measured with scanning electron microscope yields $d=1.30\pm0.03$ μm (Altshuler *et al.*, 1982; Sharvin, 1984).

Resistance is a periodic function of the magnetic flux with the period

$$\Phi_o = h/2e$$

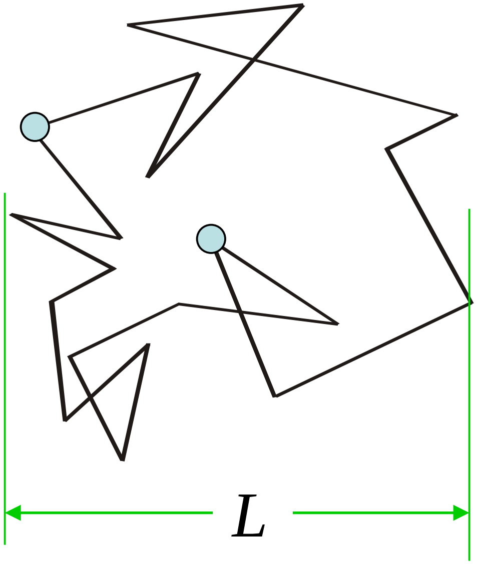
Effects of Coulomb Interaction



Electrons and disorder in solids

(Schematic guide)

ee - interaction (interference)



Ballistic regime $r \sim v_F t$

Diffusion regime $r \sim l\sqrt{t/\tau} \sim v_F\sqrt{t\tau}$

Phase $\exp(i\varphi) = \exp[i(\varepsilon_i/\hbar)t], \quad \Delta\varphi = (\Delta\varepsilon/\hbar)t$

Dephasing time

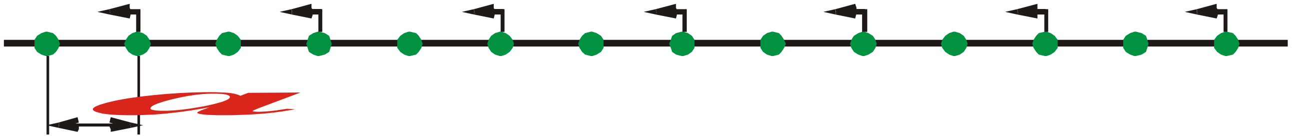
$$\left.\begin{aligned}\tau_{ee} &\sim \hbar/\Delta\varepsilon \\ \Delta\varepsilon &\sim T\end{aligned}\right\} \quad \boxed{\tau_{ee} \sim \hbar/T}$$

Dephasing length

$$\boxed{L_{ee} \approx l\sqrt{\frac{\tau_{ee}}{\tau}} \approx v_F\sqrt{\frac{\hbar\tau}{T}} \approx \sqrt{\frac{\hbar D}{T}}}$$

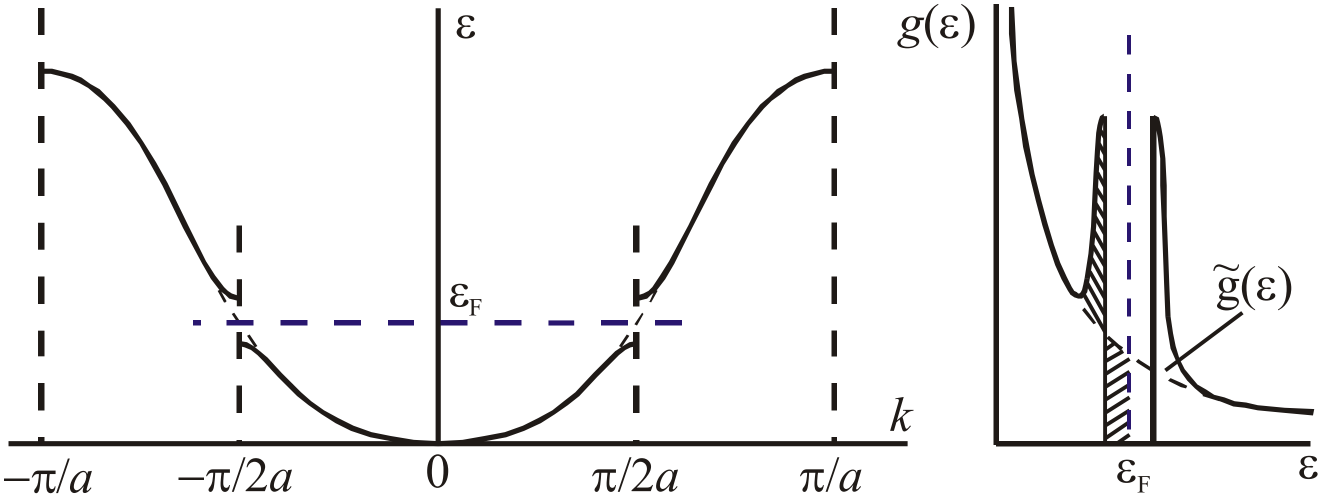
Diffusing electrons keep coherence during time τ_{ee} keeping the typical distance L_{ee}.

Peierls transition

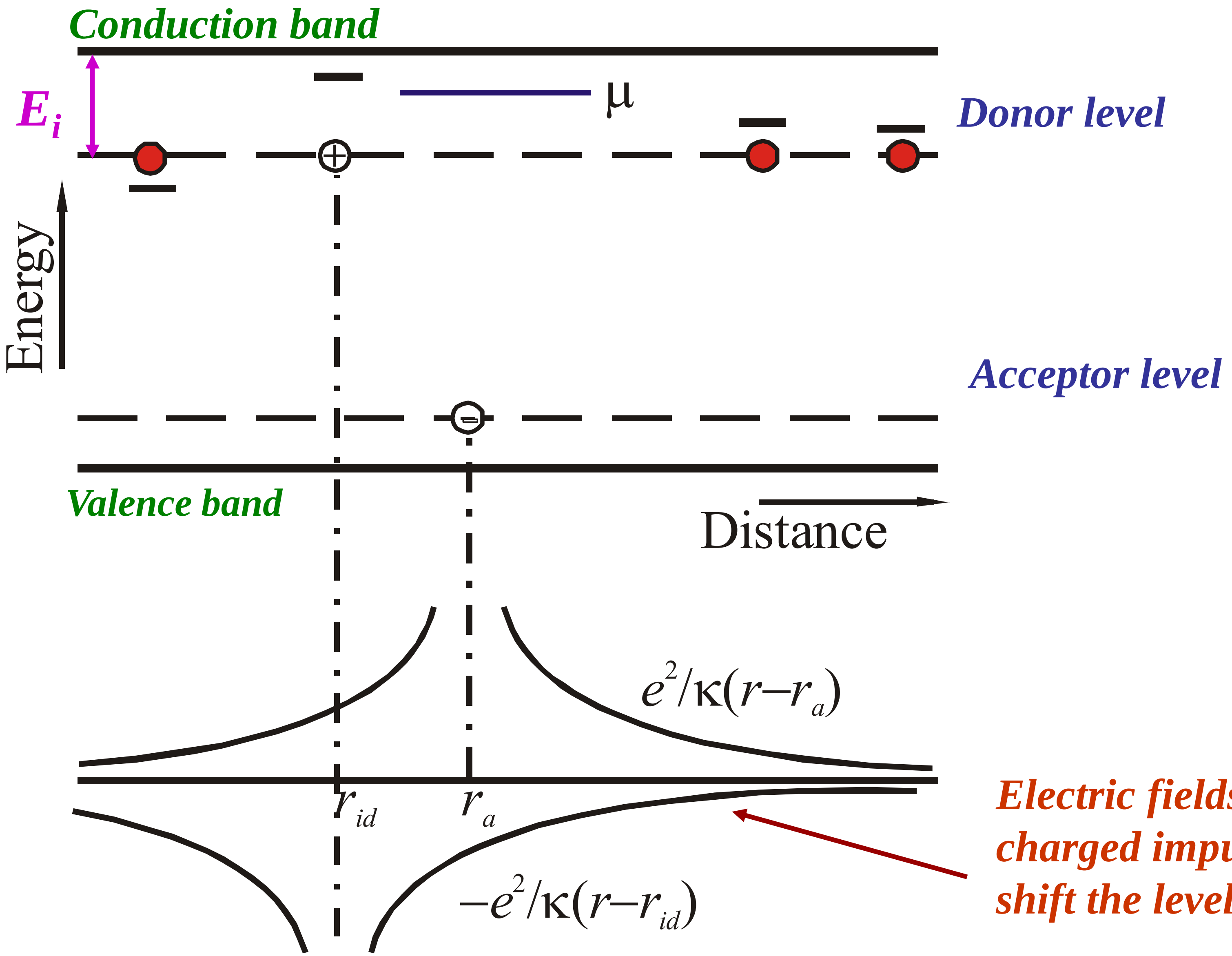


Shift of every second atom =>

period doubling a ->2a→



Impurity band



Conduction band

E_i

μ

Donor level

Energy

Acceptor level

Valence band

Distance

$e^2/\kappa(r-r_a)$

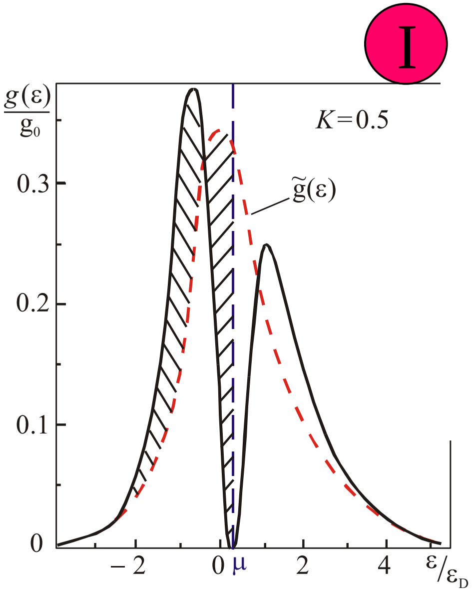
r_{id}

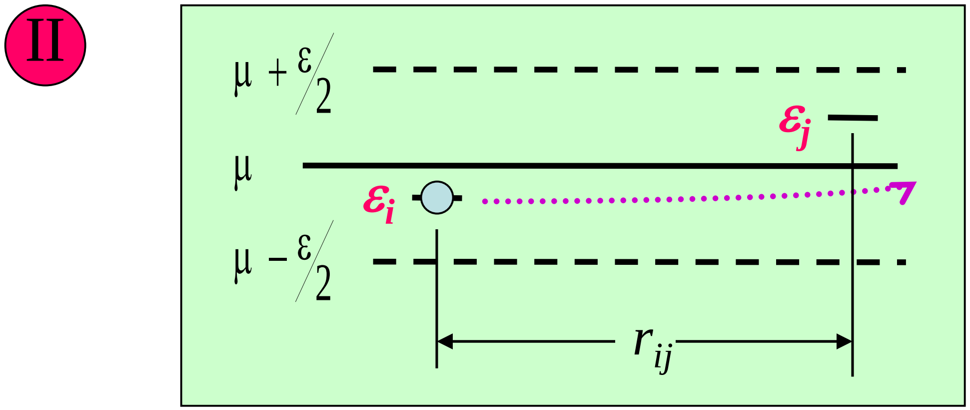
r_a

$-e^2/\kappa(r-r_{id})$

Electric fields of charged impurities shift the levels

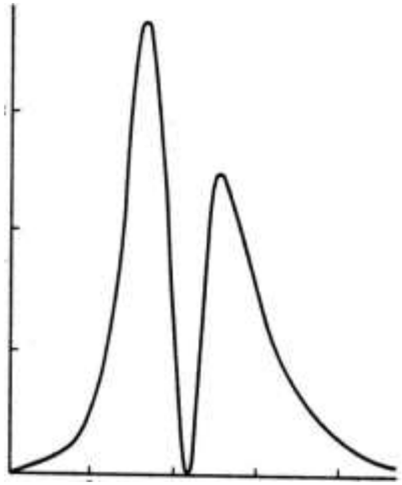
“Coulomb gap”





Energy $\varepsilon_j - e^2/\kappa r_{ij}$ ***sjhould be larger than*** ε_i

$$\varepsilon_j-\varepsilon_i-e^2/\kappa r_{ij}\ge 0 \longrightarrow e^2/\kappa r_{ij}<\varepsilon$$



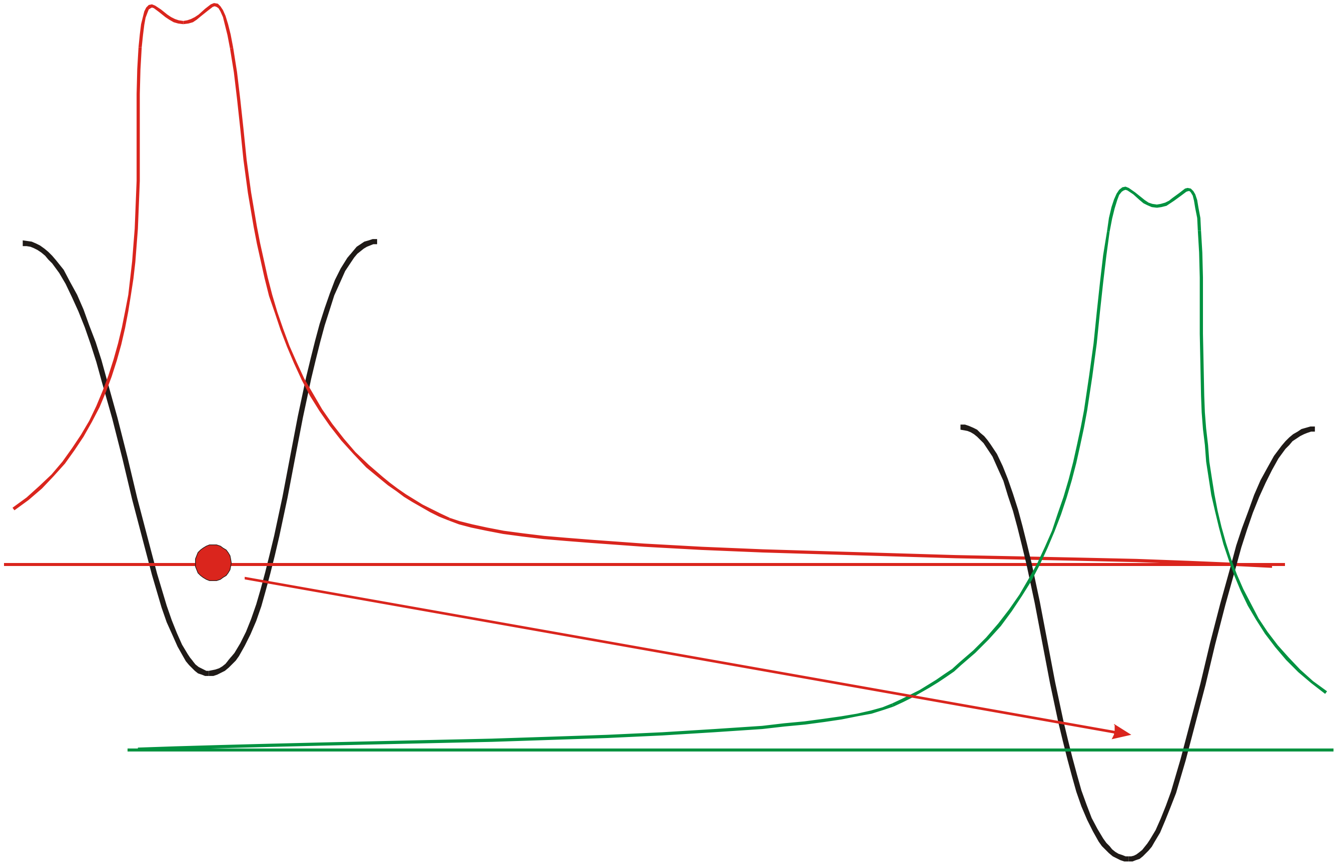
3D

$$N(\varepsilon)=r_{ij}^{-3}\le\left(\frac{\kappa\varepsilon}{e^2}\right)^3,\quad g(\varepsilon)=\frac{\partial N}{\partial\varepsilon}\propto\frac{(\varepsilon-\mu)^2\kappa^3}{e^6}$$

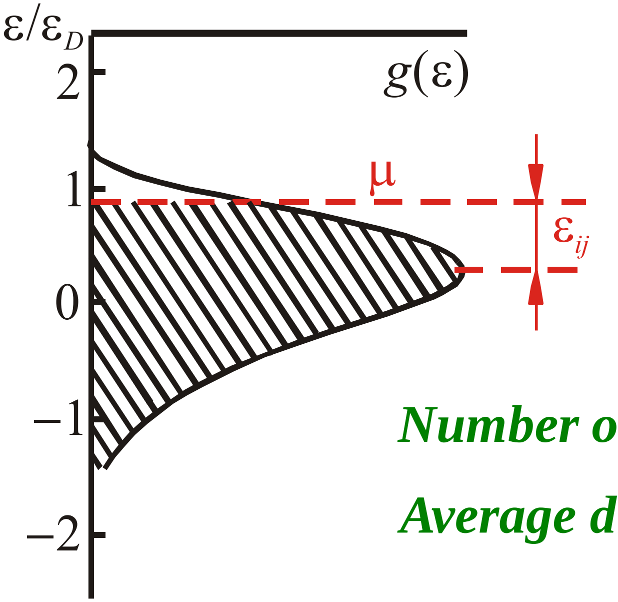
2D

$$N(\varepsilon)=r_{ij}^{-2}\le\left(\frac{\kappa\varepsilon}{e^2}\right)^2,\quad g(\varepsilon)=\frac{\partial N}{\partial\varepsilon}\propto\frac{|\varepsilon-\mu|\kappa^2}{e^4}$$

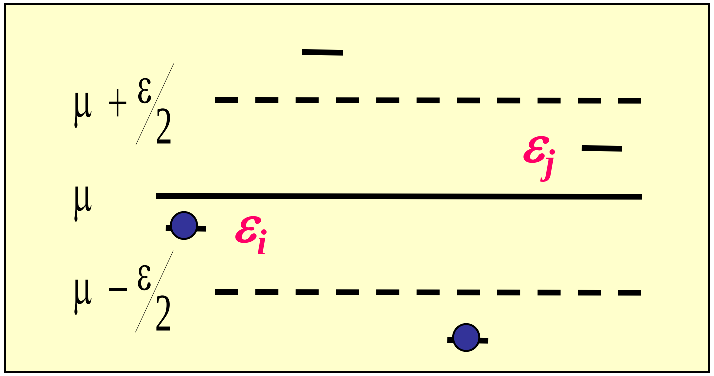
Hopping conductivity



Variable range hopping: Mott low



$$T \ll \mu \approx \varepsilon_D$$



Number of states $N(\varepsilon) = g_\mu \varepsilon$

Average distance $r_{ij}(\varepsilon) = [N(\varepsilon)]^{-1/3}$,

Energy difference $\sim \varepsilon$

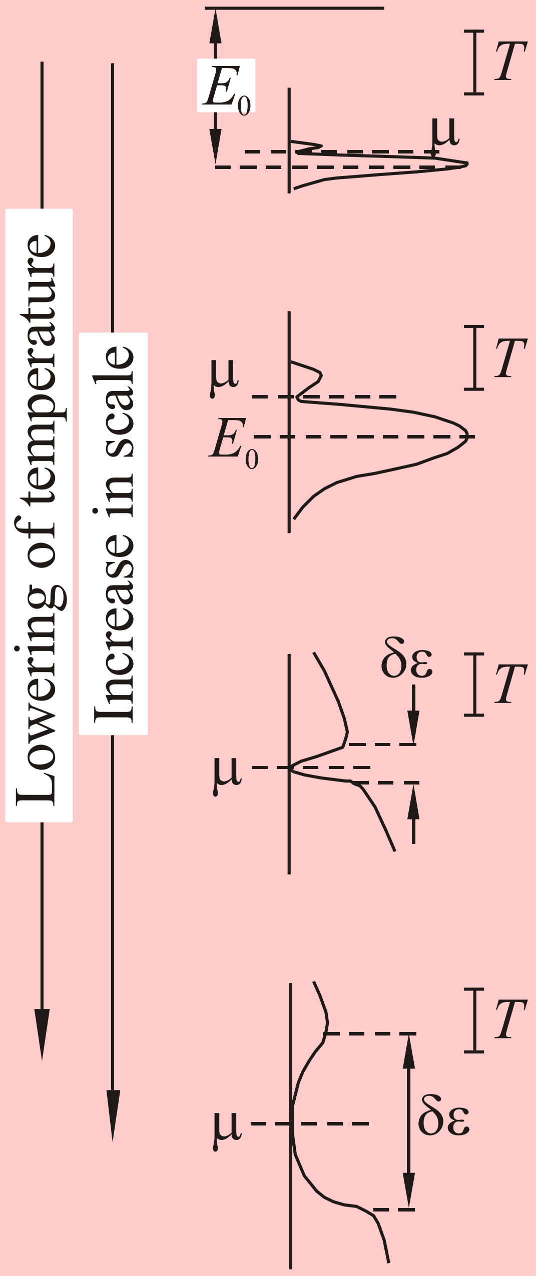
Parameter u_{ij} *of Abrahams-Miller network*

$$u_{ij} = \frac{2}{a_B[N(\varepsilon)]^{1/3}} + \frac{\varepsilon}{T} = \frac{2}{g_\mu^{1/3} a_B \varepsilon^{1/3}} + \frac{\varepsilon}{T}$$

u_{ij} *has a maximum when* $\frac{d}{d\varepsilon} u_{ij}(\varepsilon) = 0,$

=> $$\varepsilon_{\min} = \left(\frac{T}{a_B g_\mu^{1/3}}\right)^{3/4} = (T^3 T_{Mott})^{1/4}, \quad T_{Mott} = (a_B^3 g_\mu)^{-1}$$

Mechanisms of hopping conductivity



Thermal delocalization of carriers

$\exp(-E_0/T)$

Nearest-neighbor hopping

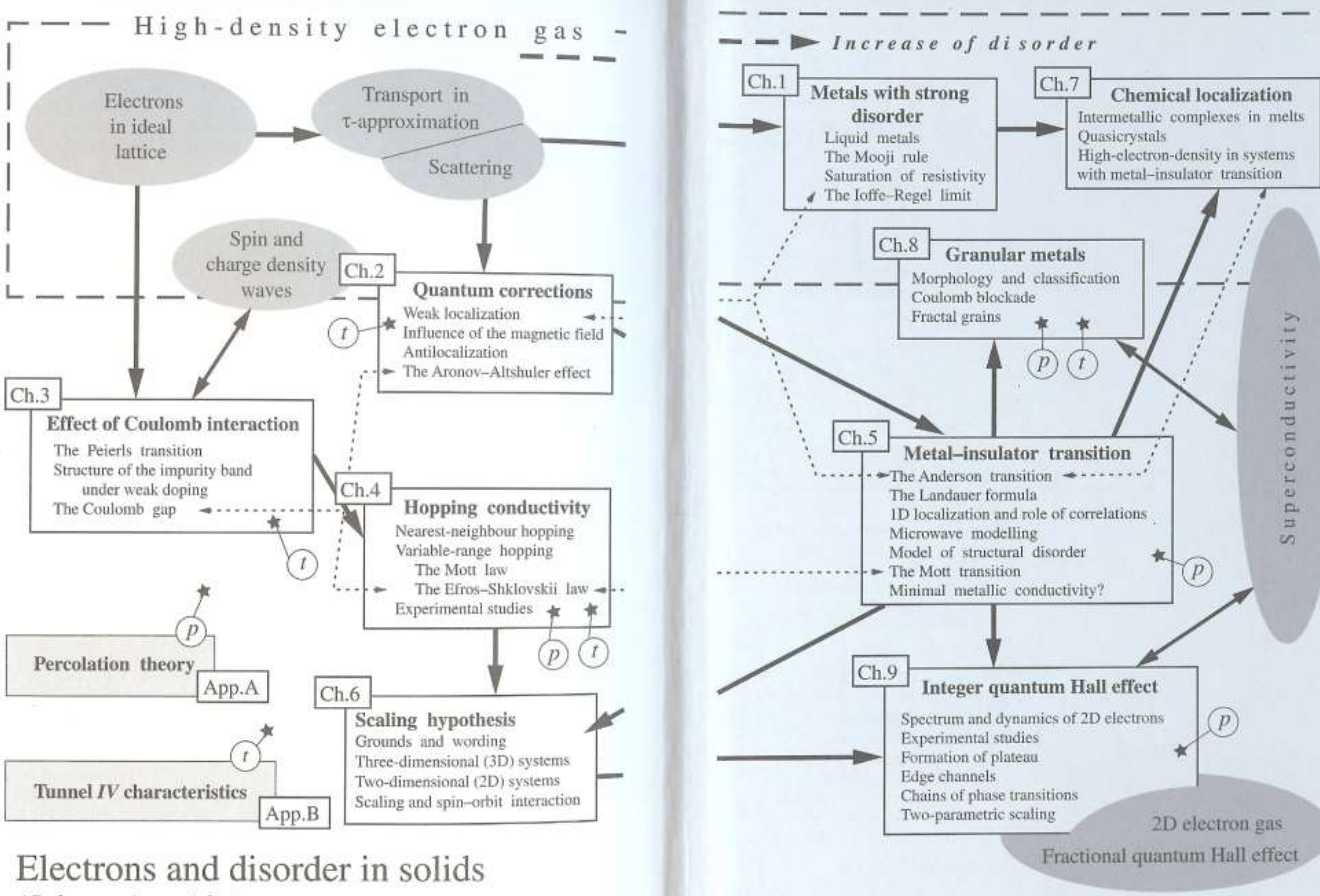
$\exp[-(\mu-E_0)/T)]$

Variable-range hopping, Mott law ($T>\delta\varepsilon$)

$\exp[-(T_M/T)^{1/4}]$

Variable-range hopping, Efros-Shklovskii law ($T<\delta\varepsilon$)

$\exp[-(T_{ES}/T)^{1/2}]$



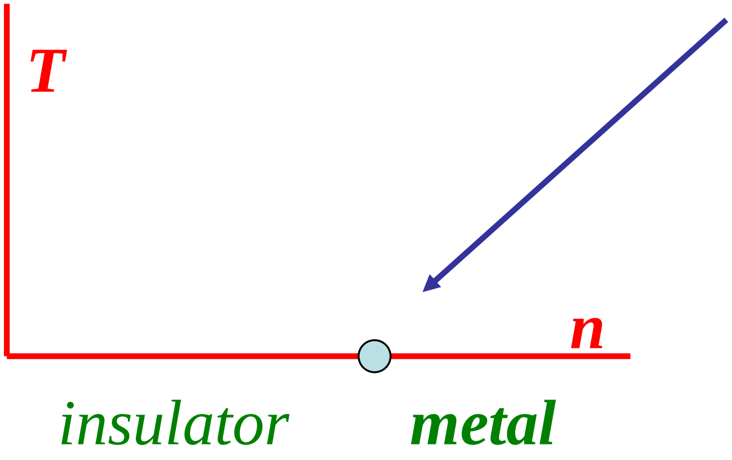
Electrons and disorder in solids
(Schematic guide)

Metal-insulator transitions

The defition: ***metal*** – $\sigma \neq 0$
insulator – $\sigma = 0$

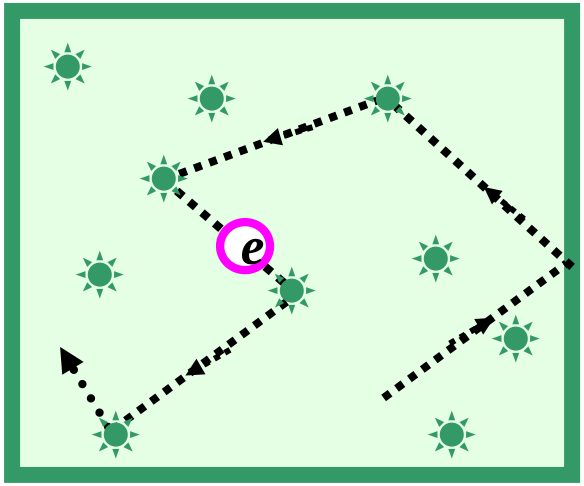
→ ***has a sense only at*** $T = 0$

Isolated point at the phase diagram



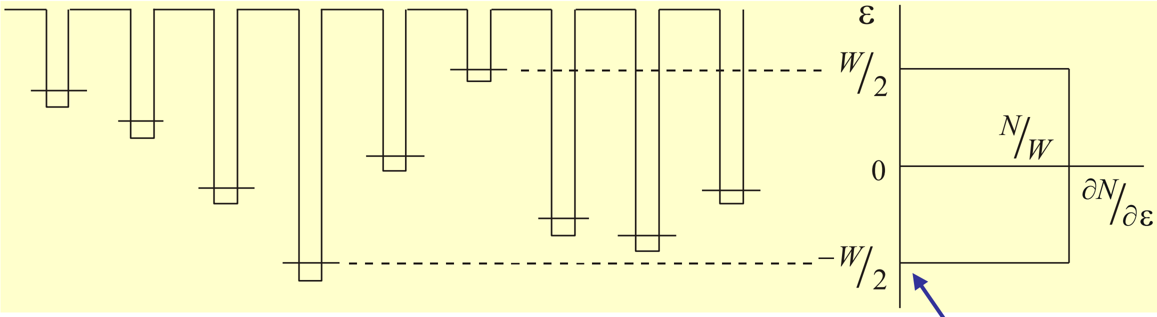
Anderson localization

Quantum particle in random quenched potential



Anderson transition

$$\frac{\partial N}{\partial \varepsilon} = \begin{cases} N/W & \text{for } |\varepsilon| < W/2 \\ 0 & \text{for } |\varepsilon| > W/2 \end{cases}$$

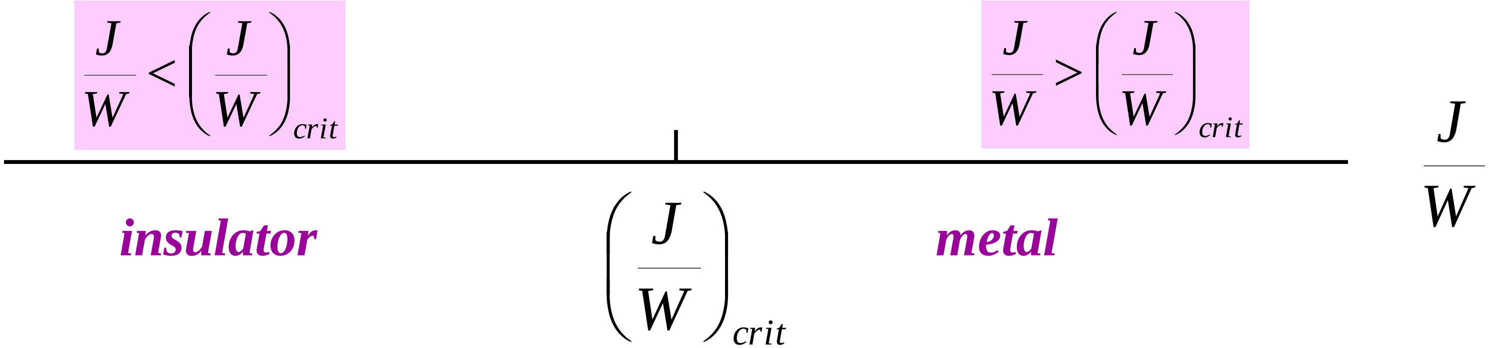


width W characterising disorder

Overlap integral

$$J = \int \psi_1^* \hat{H} \psi_2 d^3 r \propto \exp\left(-\frac{r_{12}}{a_B}\right) \equiv \exp\left(-\frac{1}{a_B n^{1/3}}\right)$$

ratio J/W - the main parameter of the problem



$$\frac{J}{W} < \left(\frac{J}{W}\right)_{crit}$$

$$\frac{J}{W} > \left(\frac{J}{W}\right)_{crit}$$

$$\frac{J}{W}$$

insulator

$$\left(\frac{J}{W}\right)_{crit}$$

metal

Delocalized states first appear at the energy band center

Mott transition

3 length scales:

$$n^{-1/3}$$

average e-e distance

$$a_B = \frac{\kappa\hbar^2}{m^* e^2}$$

Bohr radius

$$r_e = \left(\frac{4m^* e^2 n^{1/3}}{\kappa\hbar^2}\right)^{-1/2}$$

screening length

are related $r_e = \frac{1}{2}\left(a_B n^{-1/3}\right)^{1/2}$

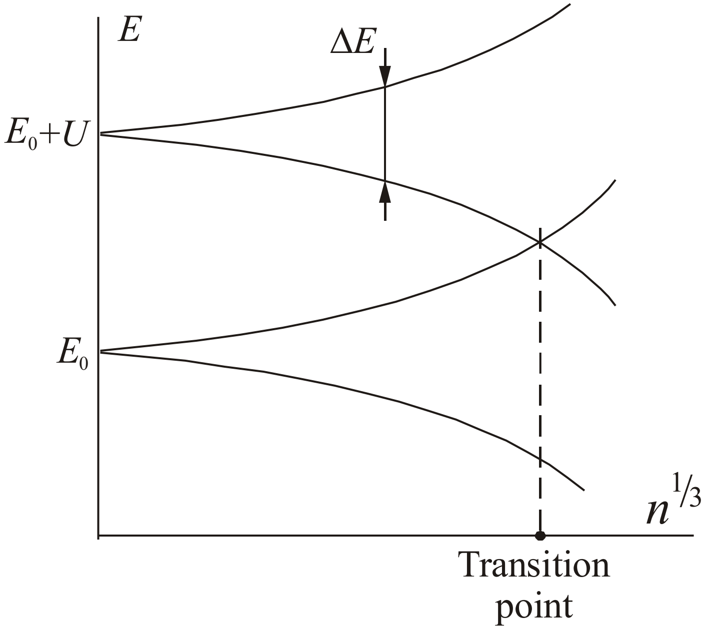
$r_e > a_B$ – *insulator*

$r_e < a_B$ – *metal*

Mott transition occurs if

$r_e = a_B$

$$a_B n_c^{1/3} = 0.25$$

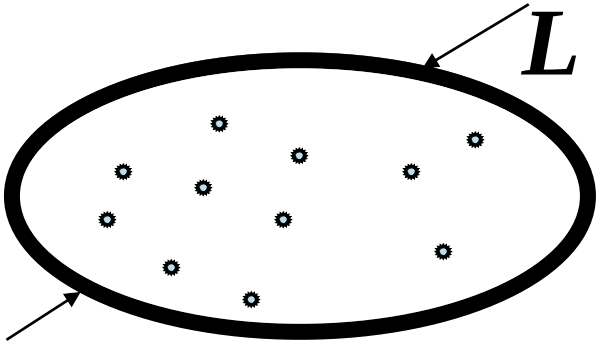


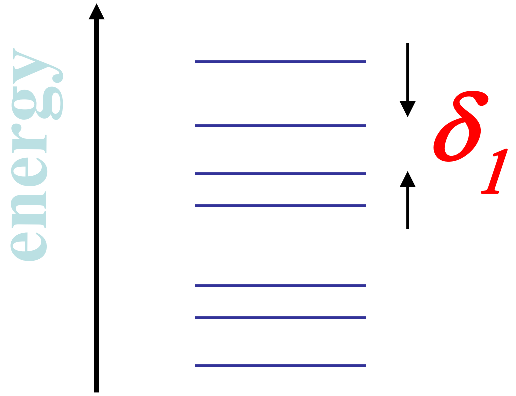
Hubbard model

Energy scales (*Thouless, 1972*)

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L is the system size;

d is the number of dimensions

2. Thouless energy

$$E_T = hD/L^2$$

D is the diffusion const

E_T has a meaning of the **inverse diffusion time** of the traveling through the system or the **escape rate** (for open systems)

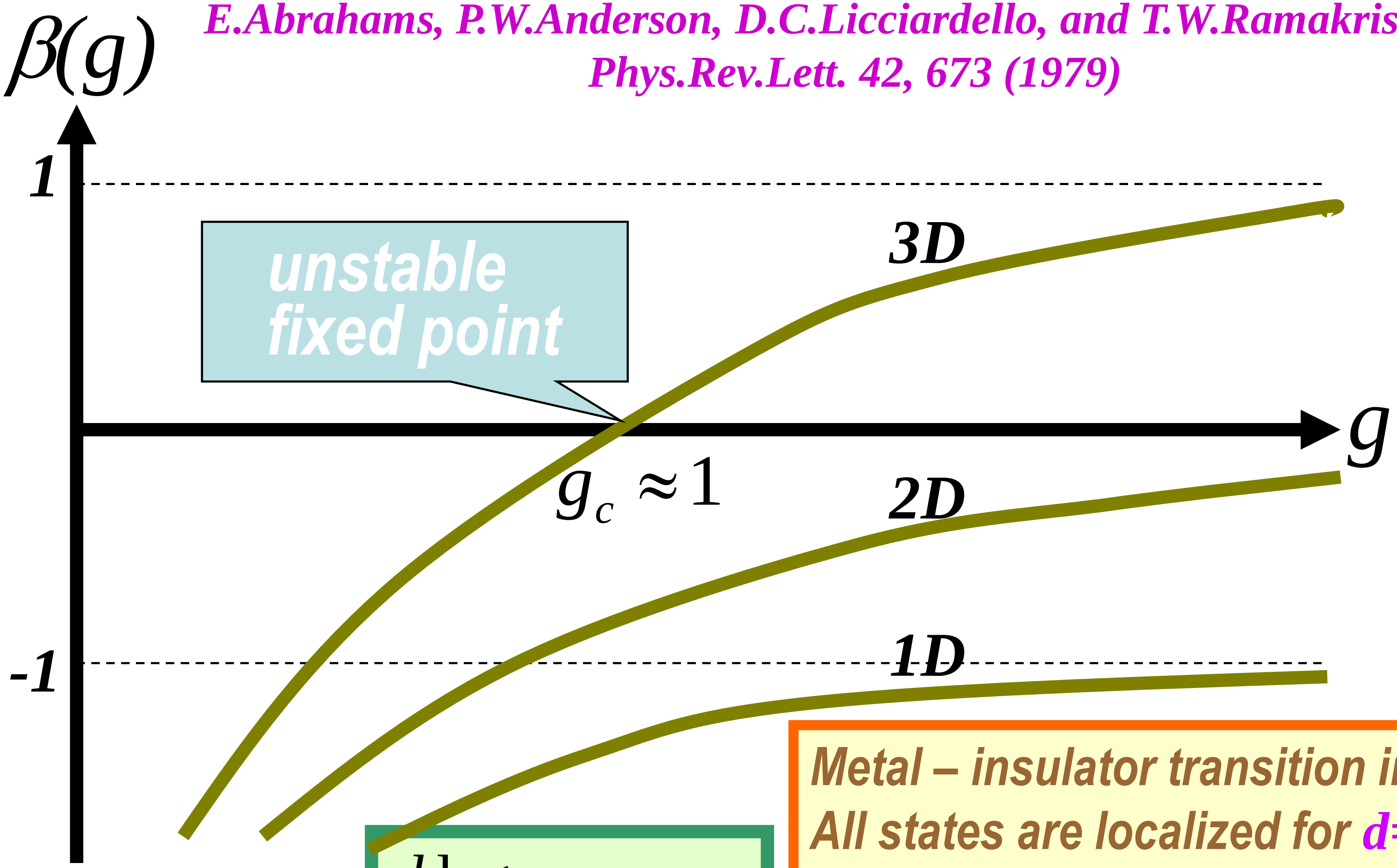
$$g = E_T / \delta_1$$

dimensionless **Thouless** conductance

$$g = Gh/e^2$$

Scaling theory of localization

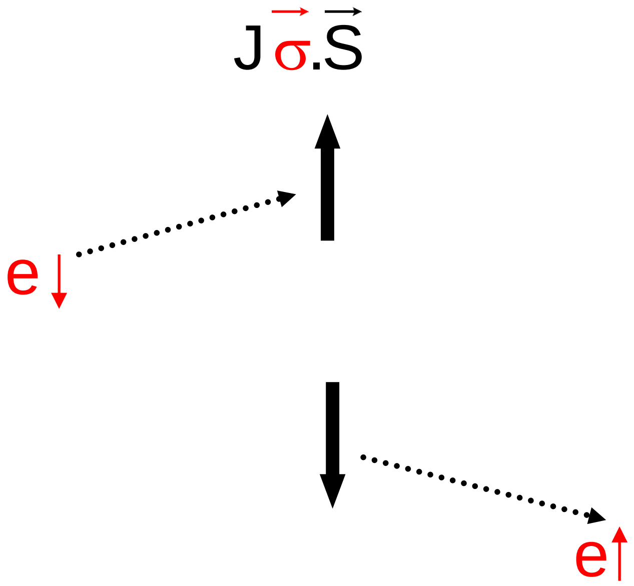
E.Abrahams, P.W.Anderson, D.C.Licciardello, and T.W.Ramakrishnan,
Phys.Rev.Lett. 42, 673 (1979)



$$\frac{d\log g}{d\log L} = \beta(g)$$

*Metal – insulator transition in **3D***
*All states are localized for **d=1,2***

Magnetic Impurities: the Kondo Effect

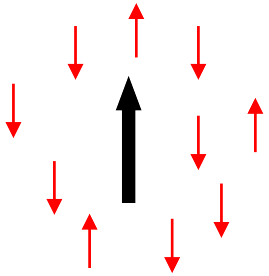


Spin-flip scattering

⇨ increased resistivity

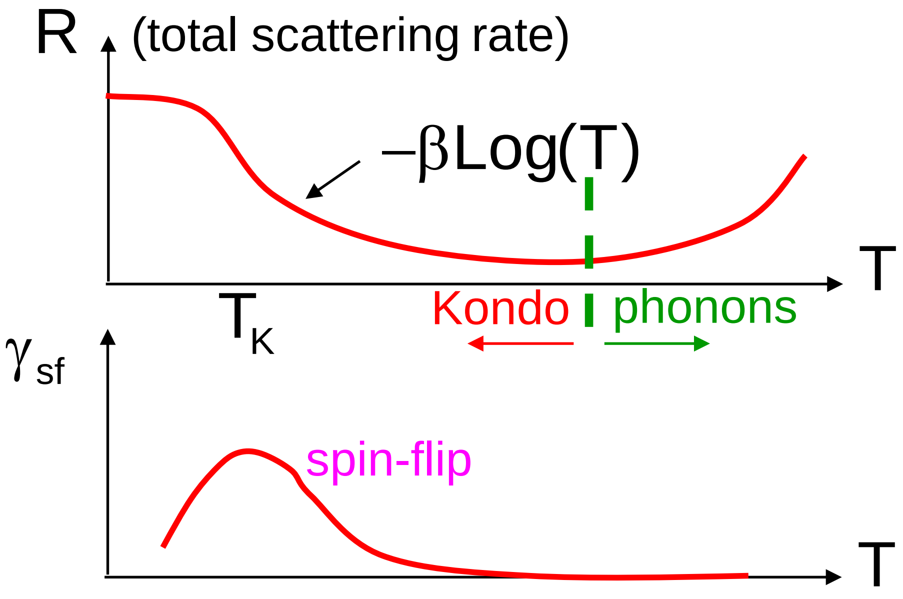
⇨ reduction of τ_ϕ

Collective effect:

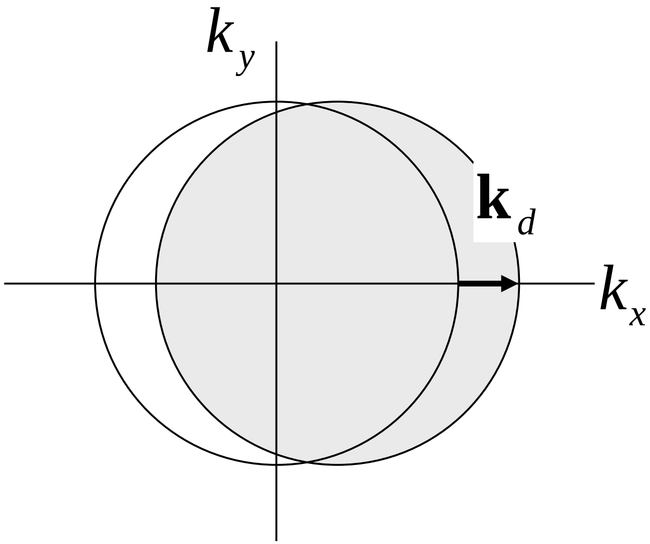


Formation of a
singlet spin state

$$k_B T_K \approx E_F e^{-1/\nu J}$$



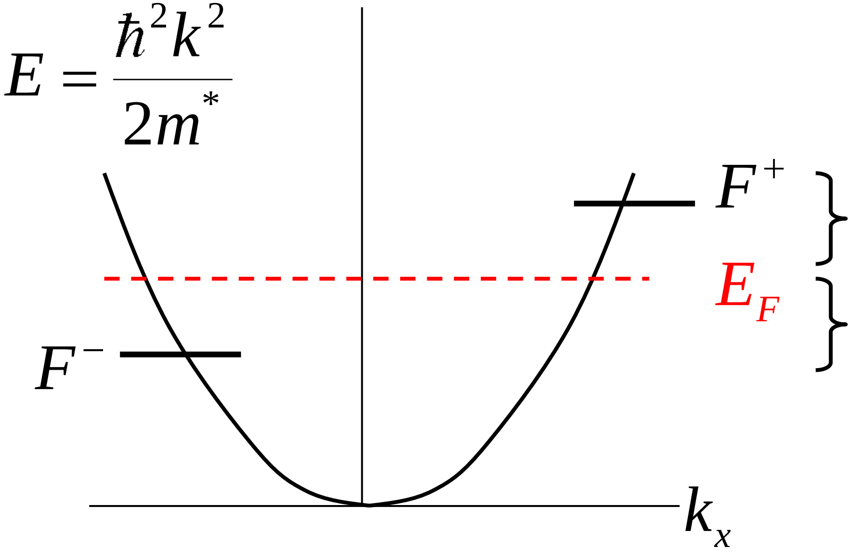
Semi-classical electron transport (Drude-Sommerfeld)



with electric field:
all electrons aquire drift velocity

$$\frac{\hbar \mathbf{k}_d}{m^*} = \mathbf{v}_d = \mu \mathbf{E}$$

$$E = \frac{\hbar^2 k^2}{2m^*}$$



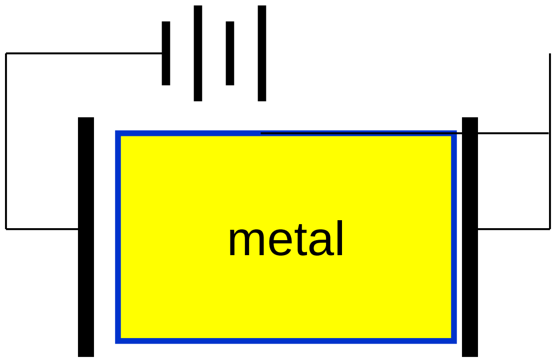
energy diagram:
states occupied with

$k_x > 0$ and $E_F < E \le F^+$

states unoccupied with

$k_x < 0$ and $F^- < E \le E_F$

Einstein relation for electric conductivity σ : conductance as a diffusion problem



$$n = n(\mu)$$

Density of electrons

Chemical potential

$$\frac{dn}{dx} = \frac{dn}{d\mu}\frac{d\mu}{dx} = eE\frac{dn}{d\mu}$$

Electric field

No current

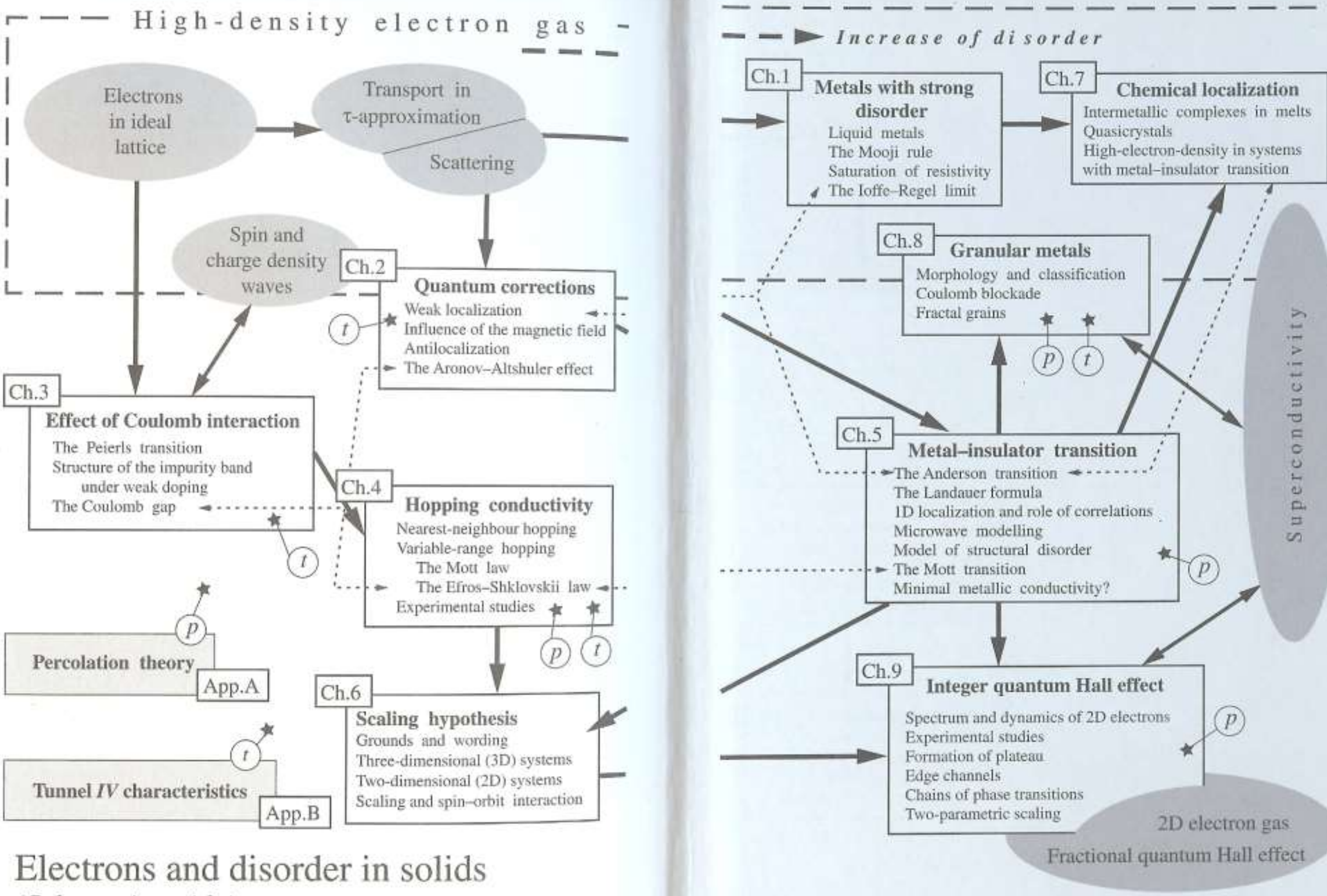
$$eD\frac{dn}{dx} = \sigma E$$

Conductivity

$$\boxed{\sigma = e^2 D\nu \qquad \nu \equiv \frac{dn}{d\mu}}$$

Density of states

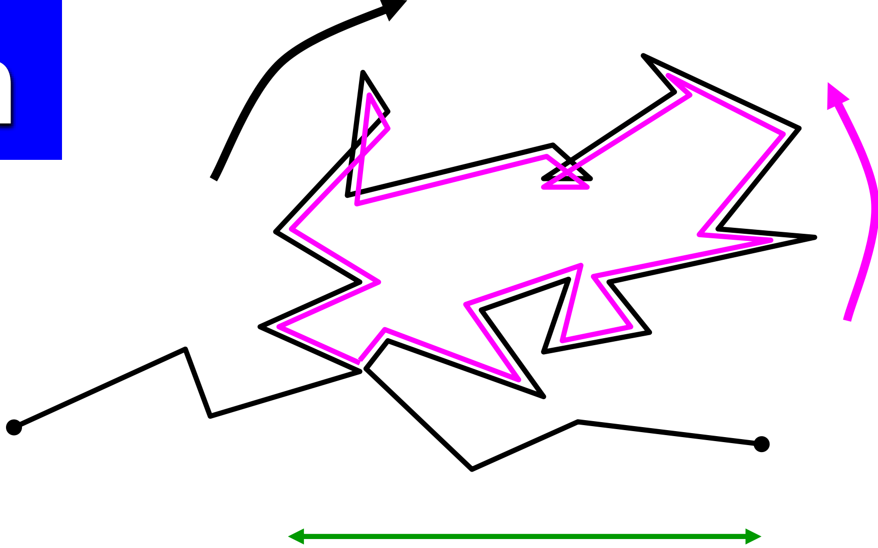
Diffusion coefficient *D* = *vl* /3



Electrons and disorder in solids

(Schematic guide)

Weak Localization



Quantum corrections to conductivity

$$\sigma = \frac{ne^2 l}{\hbar k_F}$$

Two types of electronic scattering

Elastic scattering, probability **$1/\tau$**

Inelastic scattering, probability **$1/\tau_\varphi$**

Phase φ of the wave function

$\tau \ll \tau_\varphi$ - phase coherence

$$\psi \propto \exp(i\varepsilon t/\hbar)$$

$$\varphi = \varepsilon t/\hbar \quad => \quad \delta\varphi = \delta\varepsilon\, t/\hbar$$

Semiclassical description of electric conductivity

At low temperatures conductivity saturates and has the value

$$\sigma = \frac{ne^2 l}{\hbar k_F}$$

$k_F l > 1$ **– Ioffe-Regel rule**

$$k_F = (3\pi^2 n)^{1/3}$$

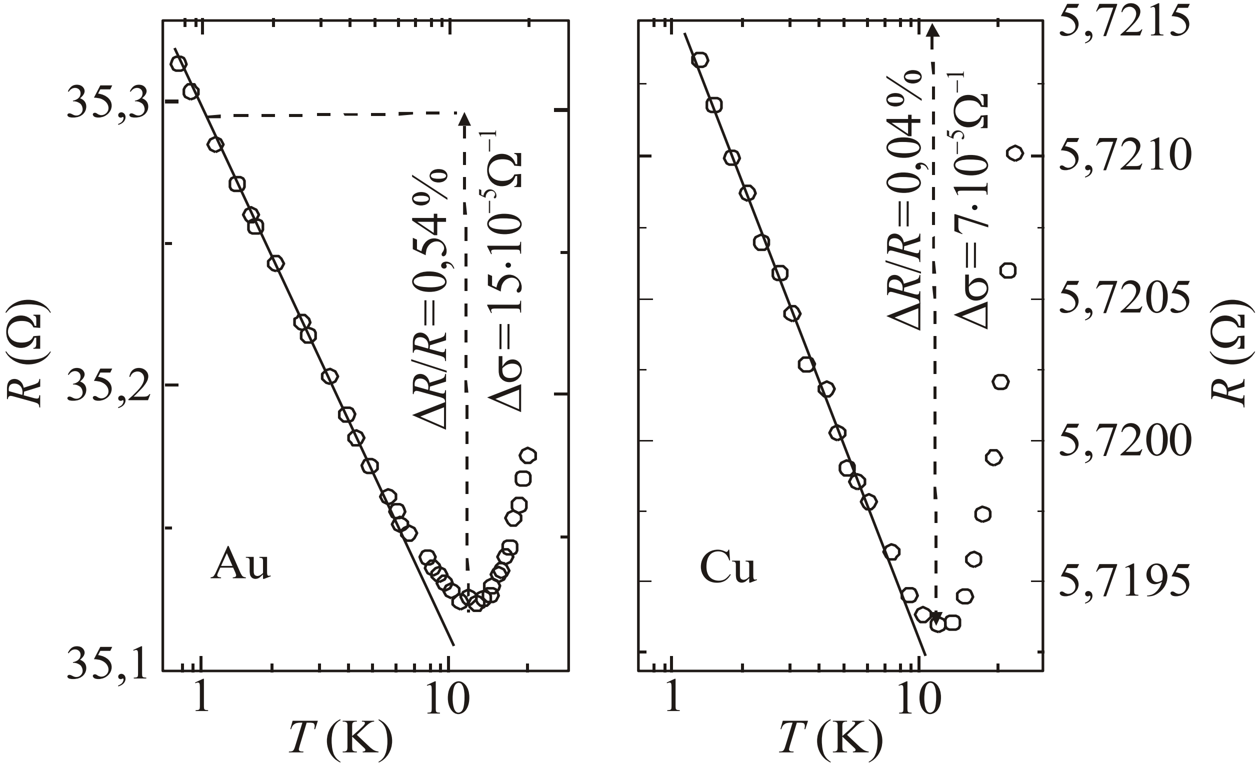
$$\left.\begin{aligned} \sigma &= A\frac{e^2}{\hbar} n^{1/3}(k_F l) \\ n^{-1/3} &\approx 3\,\mathring{A} \end{aligned}\right\}$$

$$\boxed{\rho_{max} \sim (100 \div 1000)\ \mu\Omega\cdot\text{cm}}$$

Semiclassical approach should break down for small values of l

First experiments in 1981-1982:

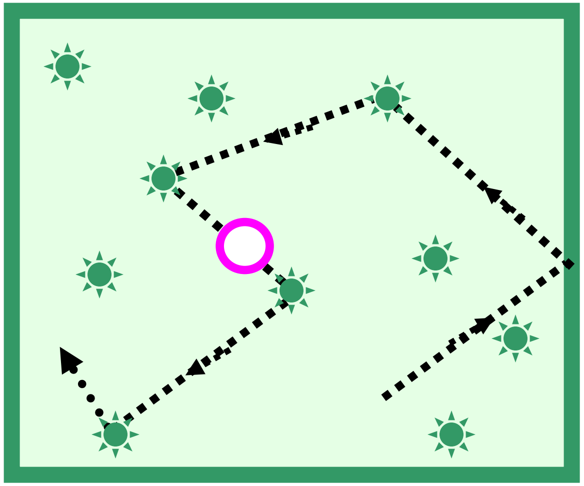
Anomalous behaviour of resistivity of disordered metallic films – no saturation at low *T*



Au – S. Dorozhkin, *et al.*, JETP Lett. **36,** 15 (1982)

Cu – van der Dreis *et al*., PRL **46,** 565 (1981)

Classical diffusion



Random walk

Density fluctuations $\rho(\mathbf{r},t)$ at a given point in space $\mathbf{r}$ and time t.

$$\frac{\partial\rho}{\partial t}-D\nabla^2\rho=0$$

Diffusion Equation

D - Diffusion constant

Mean squared distance from the original point at time t

$$\langle r(t)^2\rangle = Dt$$

Probability to come back (to the element of the volume dV centered at the original point)

$$P\left(r(t)=0\right)dV=\frac{dV}{(Dt)^{d/2}}$$

Diffusion description fails at short scales

Why?

Einstein: there is no diffusion at too short scales – there is memory, i.e., the process is not marcovian.

$$r(t) = \sqrt{Dt}$$

$$\frac{dr}{dt} = \sqrt{\frac{D}{2t}}$$

Does velocity diverge at $t \to 0$?
No because at times shorter than mean free time process is not marcovian and there is no diffusion

Quantum coherence:
there is memory at large distances

Diffusion description fails at large scales.
Why ?

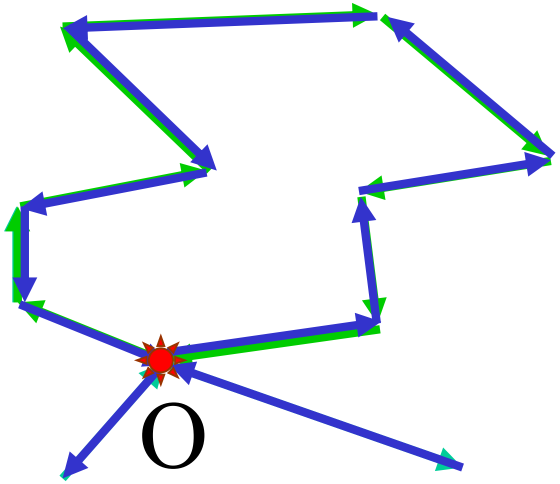
There is phase memory at large distances in quantum case

Quantum corrections at large conductance - **weak localization**

WEAK LOCALIZATION

$$\varphi = \oint \vec{p}\,d\vec{r}$$

Phase accumulated when traveling along the loop

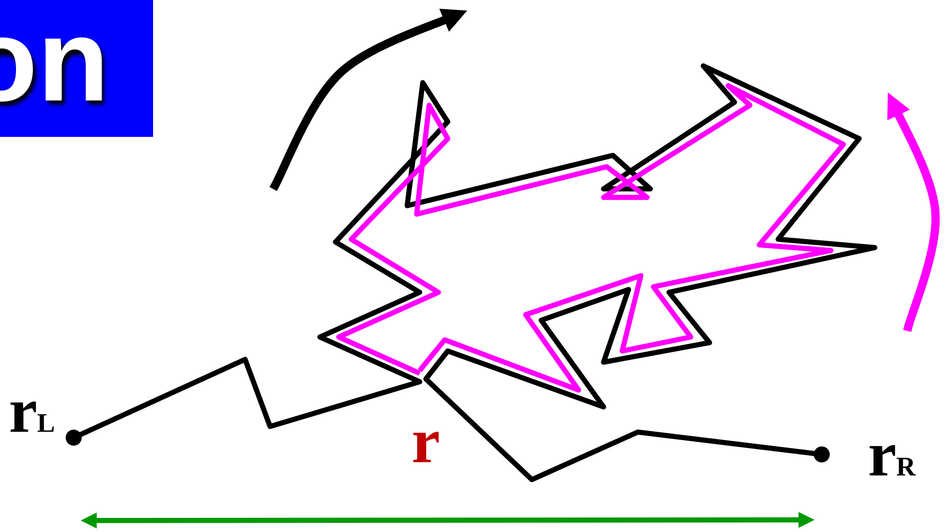


The particle can go around the loop in two directions

$$\varphi_1 = \varphi_2$$

Memory!

Weak Localization



without interference

$$|A_1|^2+|A_2|^2= 2A^2$$

$$|A_1|=|A_2|=A$$

$\tau \ll \tau_\varphi$ - phase coherence

$A_{1,2}$ are the quantum mechanical amplitudes to return to the point **r** by clockwise/ counter-clockwise propagation with equal phases $\varphi_1 = \varphi_2$

with interference

$$|A_1+A_2|^2 =$$
$$|A_1|^2+|A_2|^2 +2|A_1A_2| = 4A^2$$

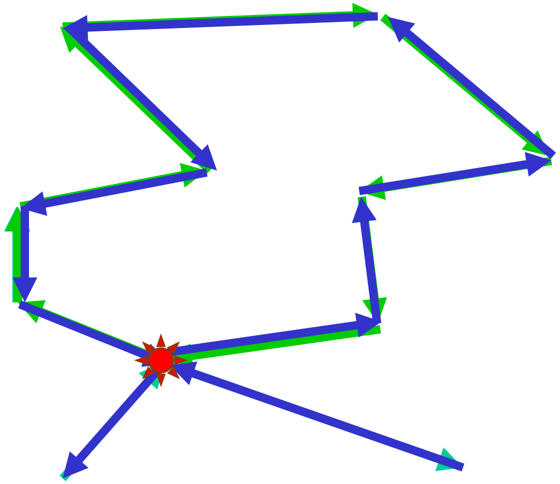
Conductance is defined by the probability of transmission from $\mathbf{r}_L$ to $\mathbf{r}_R$ (left to right)

-> Probability to return to the point **r** increases
=> Conductance is reduced

WEAK LOCALIZATION

$$\varphi = \oint \vec{p}\,d\vec{r}$$

Phase accumulated when traveling along the loop



The particle can go around the loop in two directions

$$\varphi_1 = \varphi_2$$

Constructive interference $\Rightarrow$ ***probability to return to the origin gets enhanced*** $\Rightarrow$ ***diffusion constant gets reduced.***

Tendency towards localization

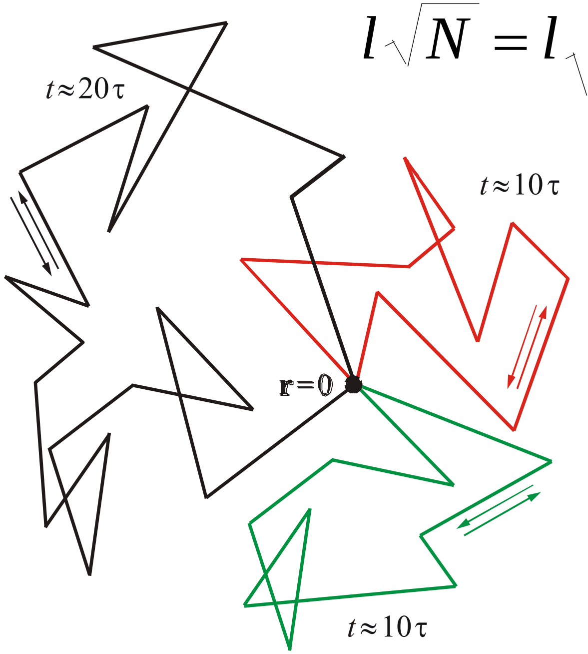
Breakdown of classical diffusion

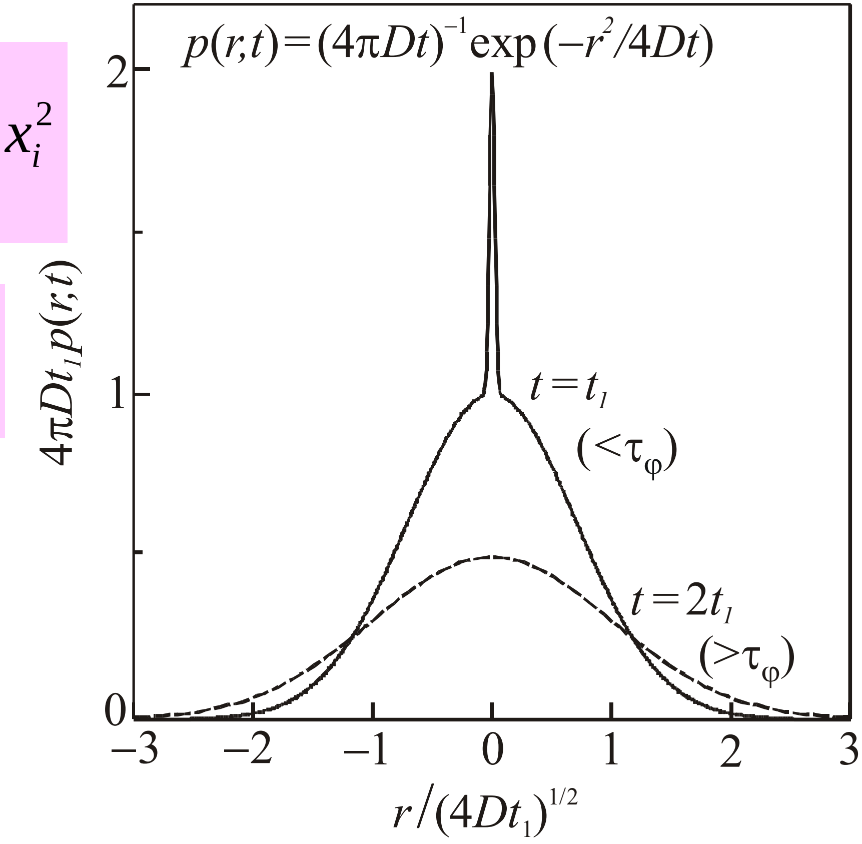
$$p(r,t) = \frac{1}{(4\pi Dt)^{d/2}} \exp\left(-\frac{r^2}{4Dt} \right), \qquad r^2 = \sum_{i=1}^{d} x_i^2$$

Diffusion coefficient $$D = \frac{1}{d} lv \approx \tau v^2 = l^2/\tau$$

Distribution width after N steps

$$l\sqrt{N} = l\sqrt{t/\tau} = \sqrt{Dt} = L_N$$





without interference $|A_1|^2+|A_2|^2 = 2A^2$

with interference $|A_1+A_2|^2 =$

$= |A_1|^2+|A_2|^2 + 2|A_1 A_2| = 4A^2$

How to estimate the correction to the conductivity ?

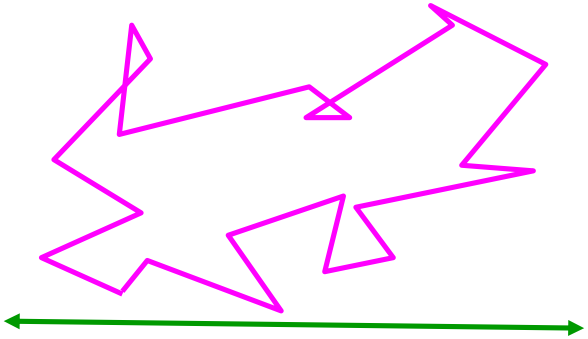
First, we introduce the concept of dimensionality:

Consider a film with thickness b

and compare the phase-breaking length L_φ with b

$b < L_\varphi$ => dimensionality $d = 2$

$b > L_\varphi$ => dimensionality $d = 3$



Typical size of a loop providing the quantum correction is L_φ

In 3D correction to the conductivity is proportional to the probability P to come back to the volume element dV :

$$P = dV/(Dt)^{3/2}$$

Since semiclassical trajectory can be viewed as "wire" with diameter of the order of the de Broglie wavelength λ, one can estimate $dV \sim v_F\lambda^2 dt$

For d = 3

$$\frac{\Delta\sigma}{\sigma} \approx -\int_{\tau}^{\tau_\varphi} \frac{v_F\lambda^2 dt}{(Dt)^{3/2}} \approx -\frac{v_F\lambda^2}{D^{3/2}}\left(\tau^{-1/2}-\tau_\varphi^{-1/2}\right) \approx \frac{1}{k_F^2 l}\left(\frac{1}{L_\varphi}-\frac{1}{l}\right)$$

Summary of main results

$d = 3$

$$\frac{\Delta\sigma}{\sigma} \approx -\int_{\tau}^{\tau_\varphi} \frac{v_F \lambda^2 dt}{(Dt)^{3/2}} \approx -\frac{v_F \lambda^2}{D^{3/2}}\left(\tau^{-1/2} - \tau_\varphi^{-1/2}\right) \approx \frac{1}{k_F^2 l}\left(\frac{1}{L_\varphi} - \frac{1}{\underset{\substack{\uparrow \\ \text{const}}}{l}}\right)$$

L_φ – *phase-breaking length:*

$$L_\varphi \approx \sqrt{D\tau_\varphi} \approx l\sqrt{N} = l(\tau_\varphi/\tau)^{1/2}$$

τ_φ – *phase-breaking time*

$d = 2$

$$\frac{\Delta\sigma}{\sigma} \approx -\int_{\tau}^{\tau_\varphi} \frac{v_F \lambda^2 dt}{(Dt)b} \approx -\frac{v_F \lambda^2}{Db} \ln \tau_\varphi/\tau$$

$d = 1$

$$\frac{\Delta\sigma}{\sigma} \approx -\int_{\tau}^{\tau_\varphi} \frac{v_F \lambda^2 dt}{(Dt)^{1/2} b^2} \approx -\frac{v_F \lambda^2}{Db^2}\left(L_\varphi - l\right)$$

Let us introduce Conductance

$$\sigma_d = \sigma b^{3-d}$$

$$\Delta\sigma_3 \approx \frac{\mathrm{e}^2}{\hbar}\left(\frac{1}{L_\varphi} - \frac{1}{l}\right)$$

$$\Delta\sigma_2 \approx -\frac{\mathrm{e}^2}{\hbar}\ln L_\varphi \big/ l$$

$$\Delta\sigma_1 \approx \frac{\mathrm{e}^2}{\hbar}(l - L_\varphi)$$

expressions for $\Delta\sigma$ do not contain n *and* σ

Exact result in 2D: $-g_s g_v \frac{e^2}{4\pi^2\hbar}\ln\left(1 + \frac{\tau_\phi}{\tau}\right)$

C. W. J. Beenakker and H. van Houte
Solid State Physics, 44, 1-228 (1991

$g_s = 2$ is *spin degeneracy*, g_v is *valley degeneracy* (relevant for semiconductors)

This review is uploaded to Canvas (see the Modul *Additional reading*)

Summary

$$\sigma = e^2 D \nu \qquad \nu \equiv \frac{dn}{d\mu}$$

Diffusion coefficient

Density of states

The origin of weak localization:

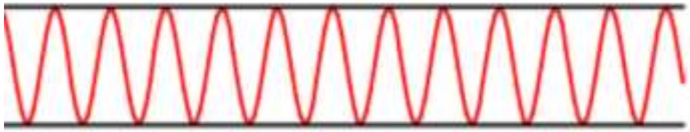
correction to the diffusion coefficient due to interference, while density of states remains unchanged

(inter-electron interactions are not taken into account)

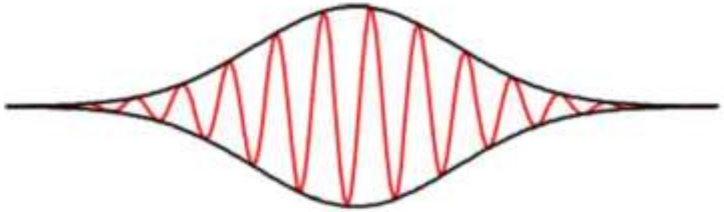
Weak Localization

Effect of Magnetic Field

Wave packets and uncertainty principle



The real part (red) and envelope (black) of an example wave.



Wave packet after integration over momentum k

Heisenberg uncertainty relation:

$$\Delta p \cdot \Delta x \geq \hbar / 2$$

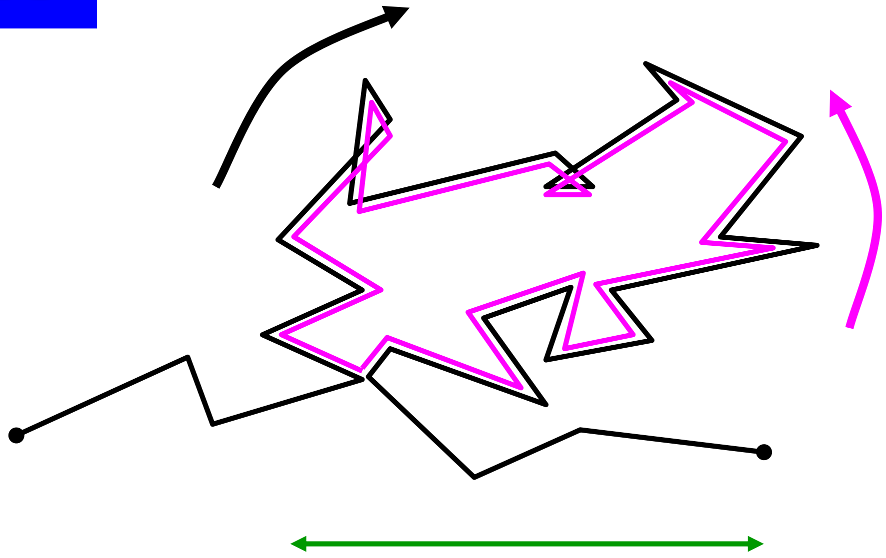
Group velocity

$$v_g = \frac{1}{\hbar}\frac{\partial E}{\partial k}$$

Effective mass

$$m^* = \hbar^2\left(\frac{\partial^2 E}{\partial k^2}\right)^{-1} = \hbar\left(\frac{\partial v_g}{\partial k}\right)^{-1}$$

Weak Localization



without interference

$$|A_1|^2+|A_2|^2= 2A^2$$

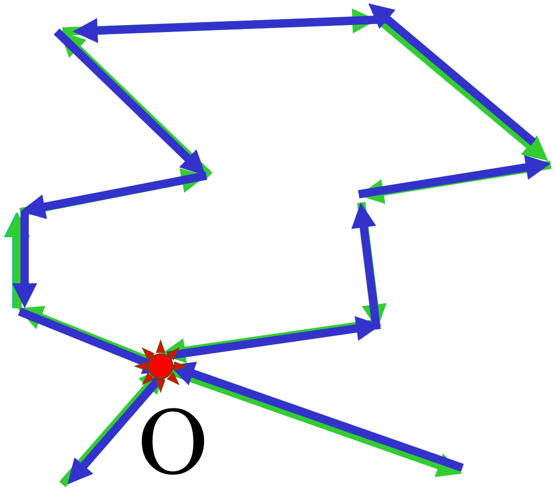
$$|A_1|=|A_2|=A$$

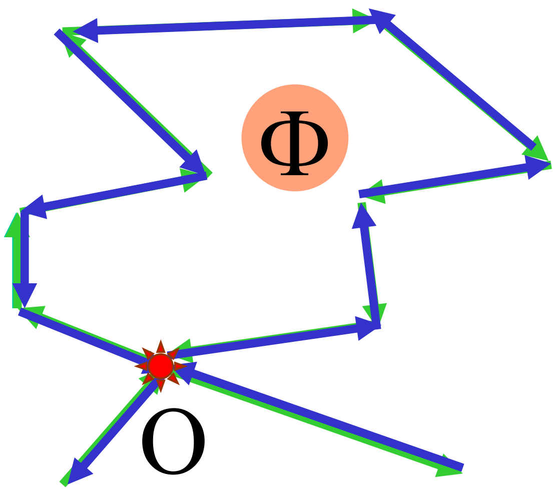
$\tau << \tau_\varphi$ - phase coherence

with interference

$$|A_1+A_2|^2 = |A_1|^2+|A_2|^2+2|A_1A_2| = 4A^2$$

Magnetoresistance





No magnetic field

$$\varphi_1 = \varphi_2$$

***With magnetic field* B**

$$\varphi_1 - \varphi_2 = 2\pi\,\Phi/\Phi_0$$

$$|A_1 + A_2|^2 = |A_1|^2 + |A_2|^2 + 2|A_1 A_2|\cos\varphi = 2A^2(1 + \cos\varphi)$$

$$|A_1| = |A_2|$$

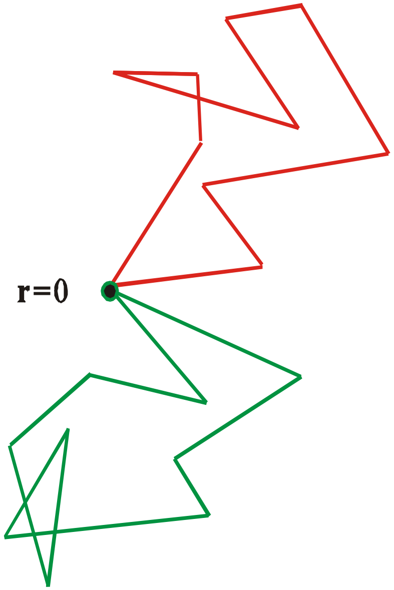
Breaking weak localization by magnetic field

low field regime *$\Omega\tau \ll 1$,*   *$\Omega = eB/m$* is the Larmor frequency

going around trajectory of area ***S***

$$\Psi \to \Psi \exp\left(i\frac{e}{\hbar}\int \mathbf{A}d\mathbf{l} \right) = \Psi \exp\left(\pm \frac{i\pi BS}{\Phi_0} \right), \quad \Phi_0 = {\pi\hbar}/{e} = h/2e$$

Phase difference **$\varphi = 2\pi\,(BS/\Phi_0)$**



<u>all diffusive trajectories have different areas S</u>

$\Rightarrow$ *weak localization is destroyed.*

Average area $\overline{S}$ *and flux* $B\overline{S}$ *depend on time*

$$B\overline{S} \approx B\overline{r^2} \approx BDt$$

How to estimate the “breaking magnetic field”

Phase difference $\varphi = 2\pi\,(BS/\Phi_0)$

We use $B\overline{S} \approx B\overline{r}^{\,2} \approx BDt$ *and replace t by* τ_φ

Since $D\tau_\varphi = L_\varphi^2$

one can formulate the condition of weak localization breaking

$$\varphi = 2\pi\,(BL_\varphi^2/\Phi_0) \simeq 1$$

Breaking field

$$B_\varphi = \frac{\Phi_0}{\pi L_\varphi^2} = \frac{\hbar}{e}(D\tau_\varphi)^{-1}$$

Magnetic length and magnetic time

$$l_B = \left(\hbar/2eB\right)^{1/2}$$

$$\tau_B = \frac{l_B^2}{D} \approx \frac{\Phi_0}{BD} \approx \Omega^{-1}(k_F l)^{-1}$$

Breaking weak localization by magnetic field

In strong magnetic field

$$\int_{\tau}^{\tau_\varphi} \rightarrow \int_{\tau}^{\tau_B}$$

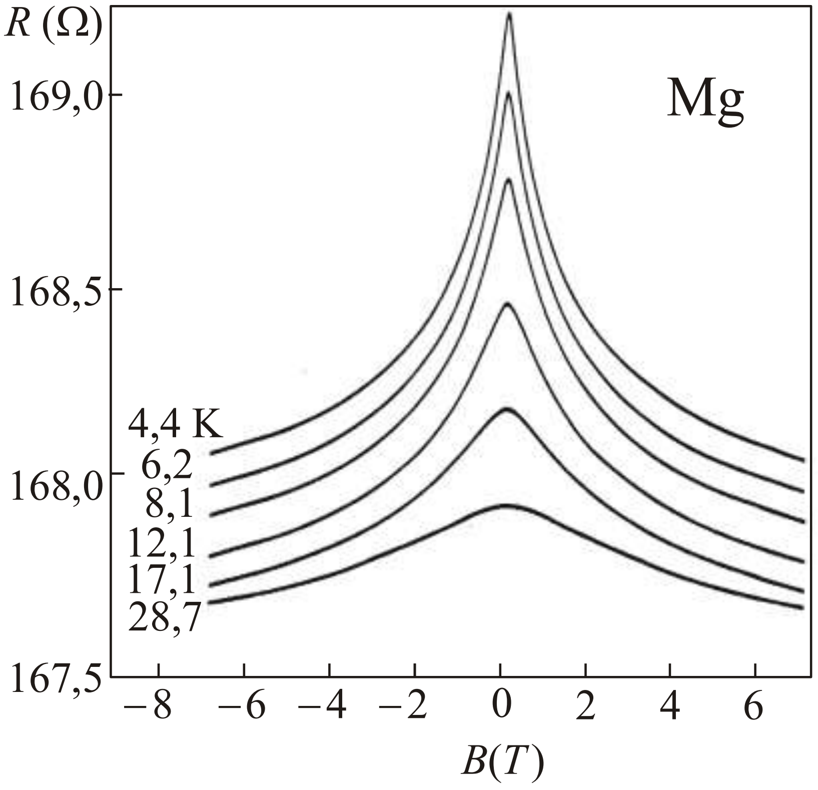
in 2 dimensions

$$\Delta\sigma(B) - \Delta\sigma(0) \sim \frac{e^2}{\hbar} \ln \frac{L_\varphi}{l_B}$$

$$l << l_B \leq L_\varphi$$

Magnetic length

$$l_B = (\Phi_0 / B)^{1/2}$$



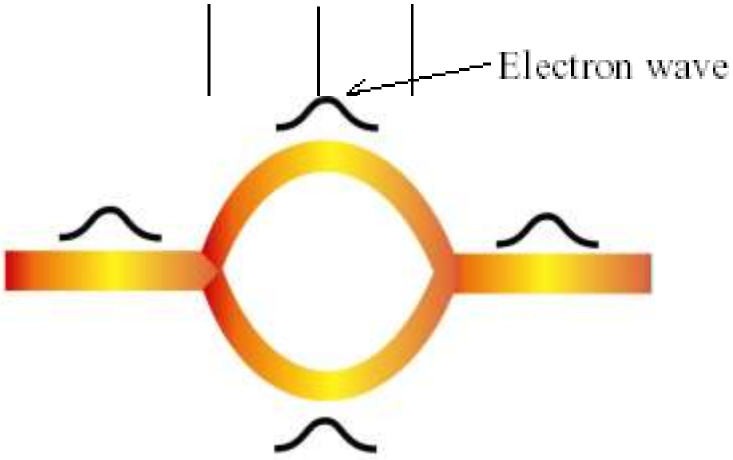
G.Bergmann, Phys.Rep. **107,** 1 (1981)

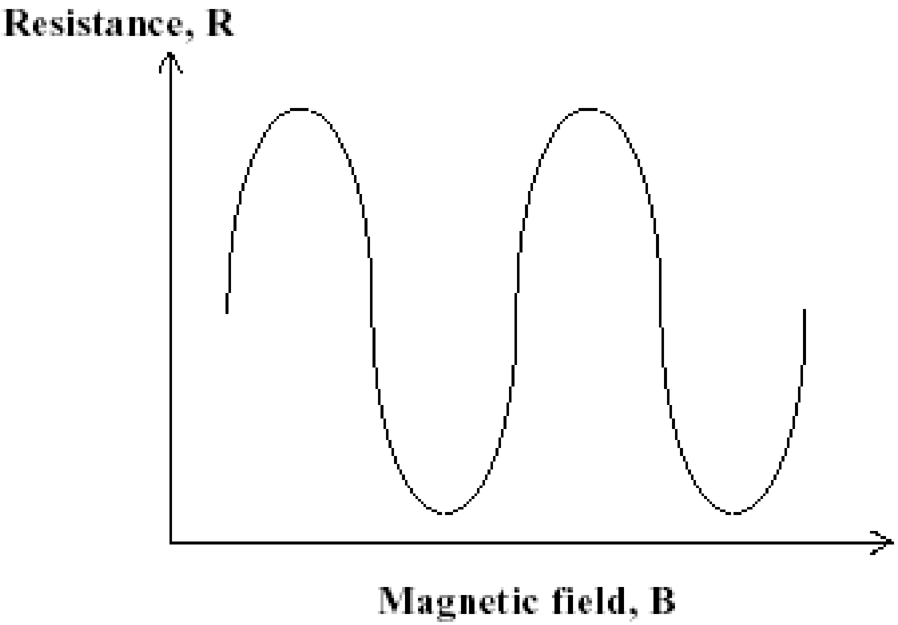
Two characteristic fields:

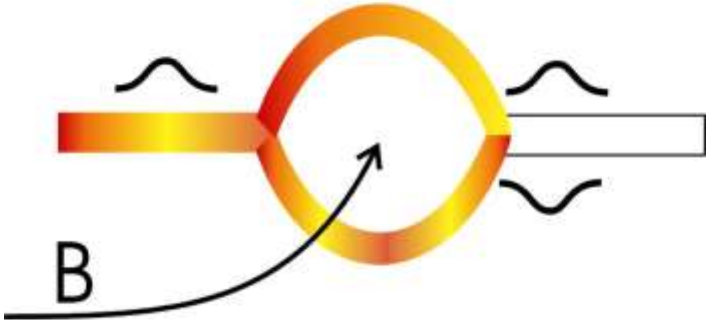
$B_\varphi \simeq \Phi_0 / L_\varphi^2$

$B_l \simeq \Phi_0 / l^2$

Aharonov-Bohm interference effects







The particle can go around the loop in two directions

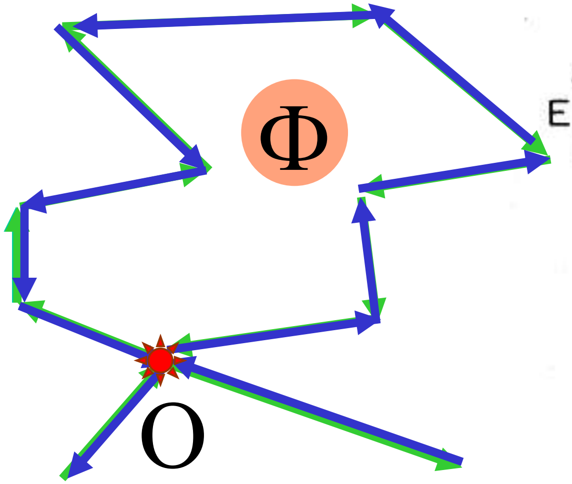
Phase accumulated when traveling along the loop

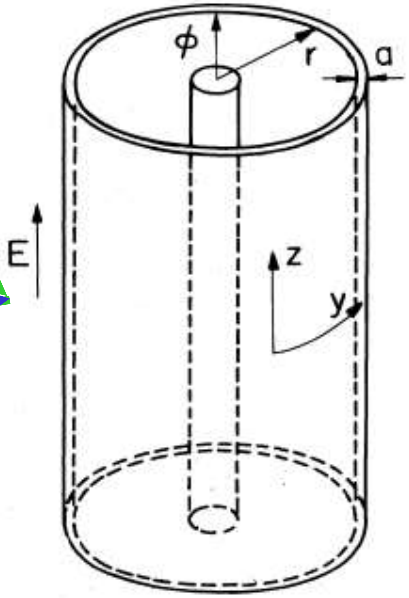
$$\varphi = \oint(\vec{p} - \vec{e}\vec{A})d\vec{r}$$

Resistance is a periodic function of the magnetic flux with the period

$$\Phi_o = h/e$$

Aharonov-Bohm effect in the WL regime





With magnetic field H

$$\varphi_1 - \varphi_2 = 2\pi\, \Phi/\Phi_0$$

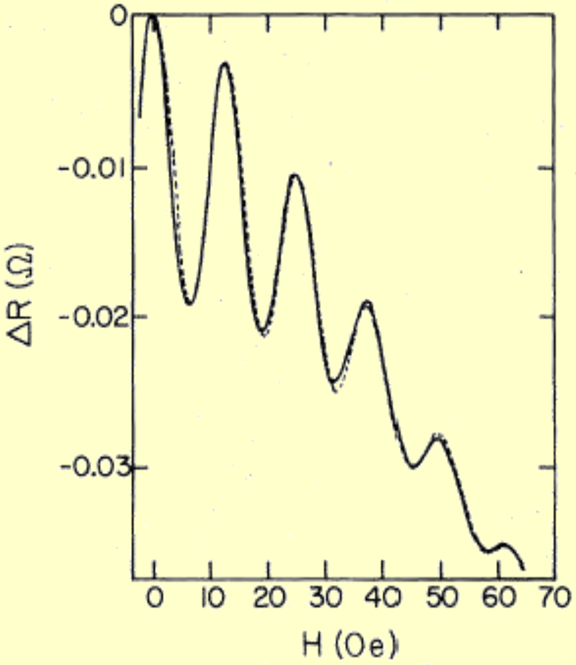
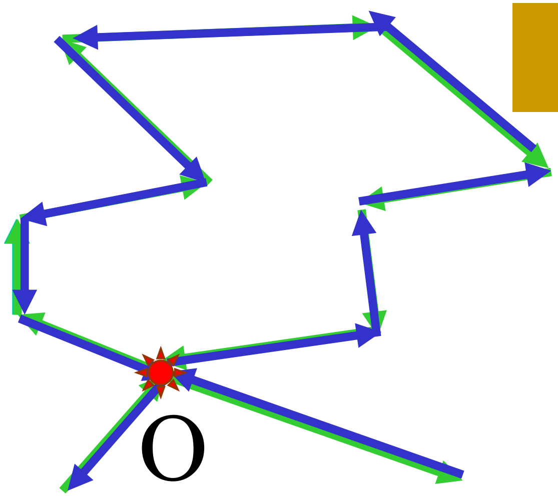


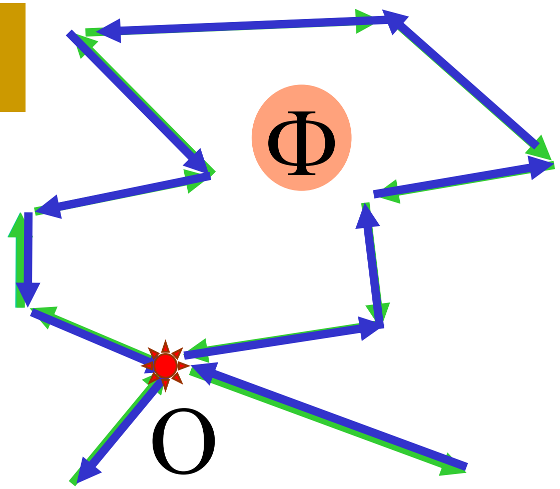
FIG. 8. Longitudinal magnetoresistance $\Delta R(H)$ at $T=1.1$ K for a cylindrical lithium film evaporated onto a 1-cm-long quartz filament. $R_{4.2}=2$ kΩ, $R_{300}/R_{4.2}=2.8$. Solid line: averaged from four experimental curves. Dashed line: calculated for $L_\varphi=2.2$ μm, $\tau_\varphi/\tau_{so}=0$, filament diameter $d=1.31$ μm, film thickness 127 nm. Filament diameter measured with scanning electron microscope yields $d=1.30\pm0.03$ μm (Altshuler *et al.*, 1982; Sharvin, 1984).

Resistance is a periodic function of the magnetic flux with the period

$$\Phi_o = h/2e$$

Magnetoresistance



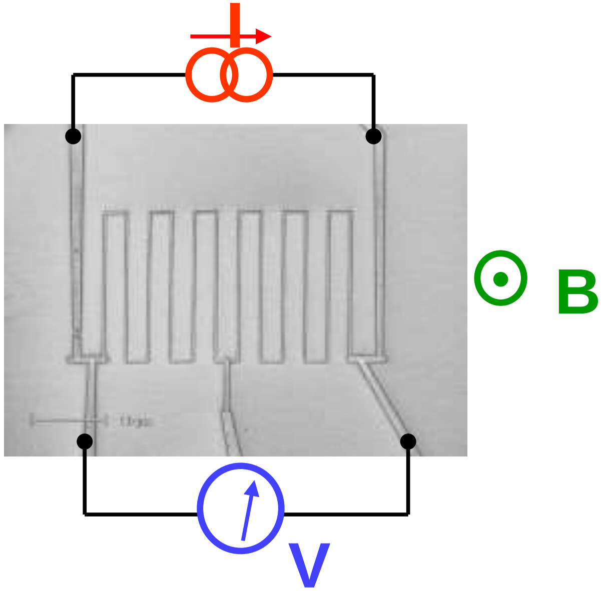


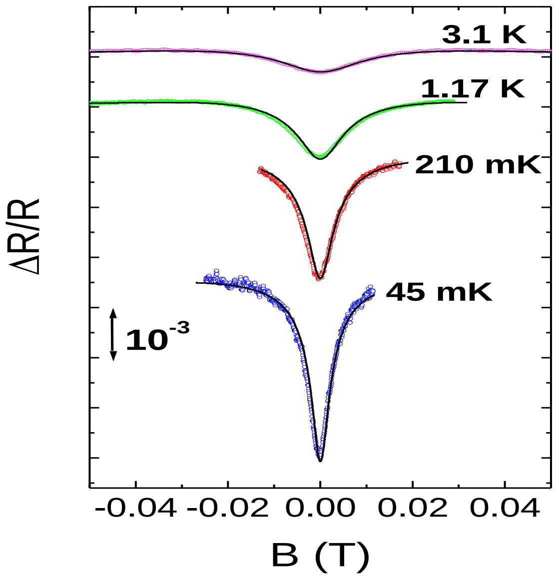
No magnetic field

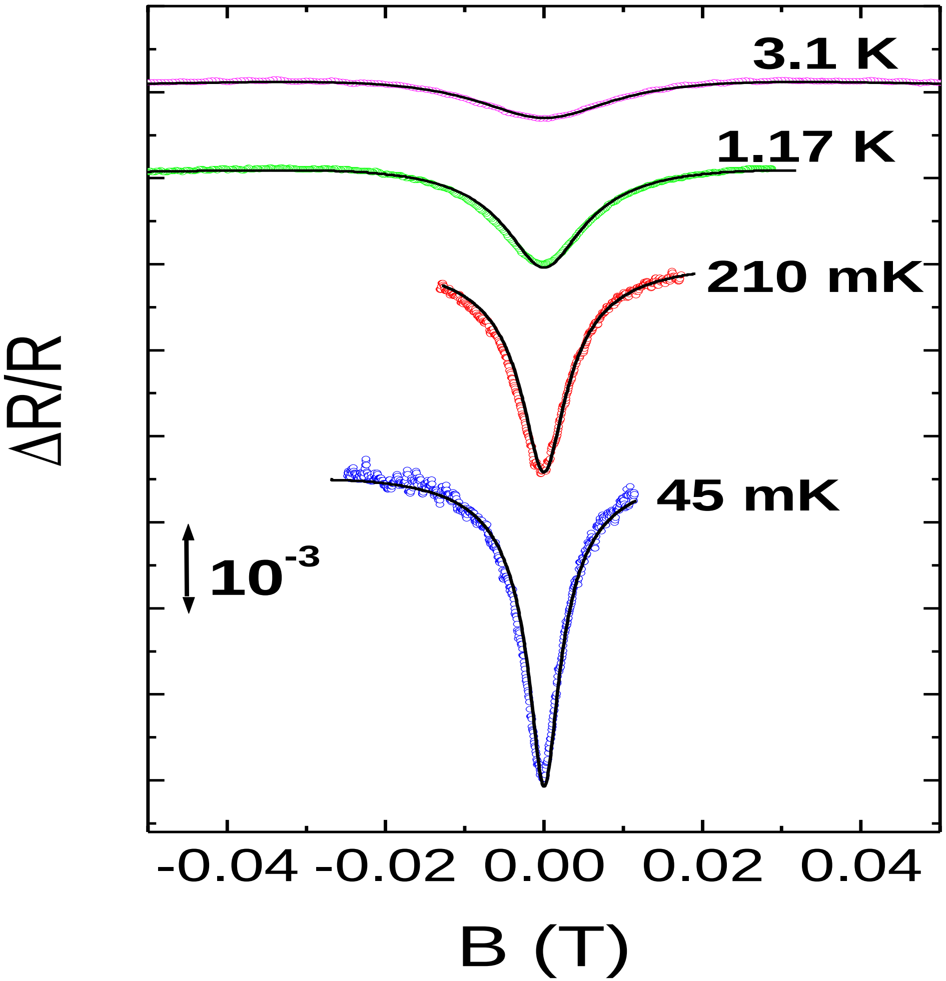
$$\varphi_1 = \varphi_2$$

With magnetic field B

$$\varphi_1 - \varphi_2 = 2\pi\, \Phi / \Phi_0$$







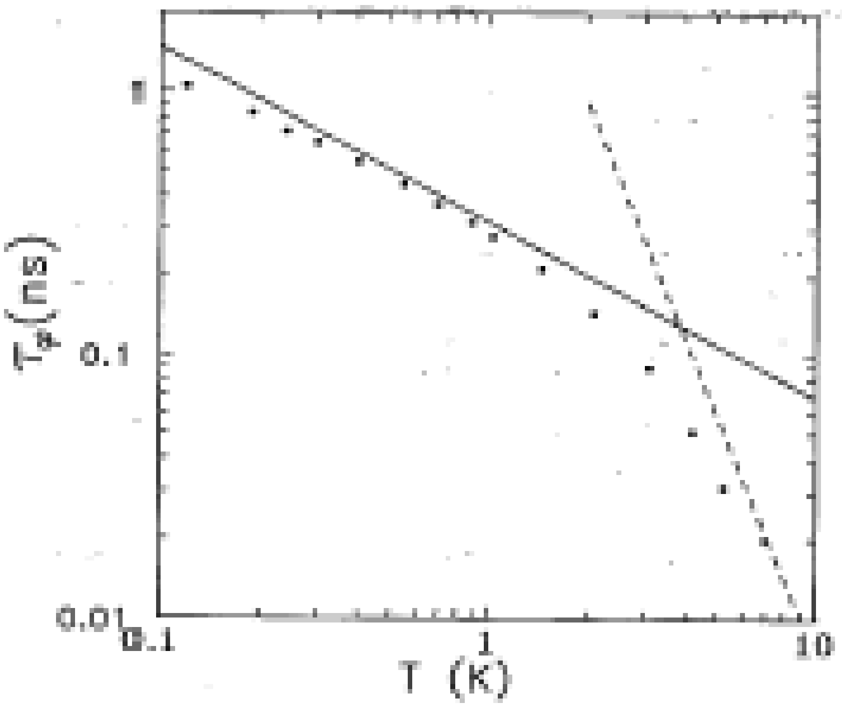
$$\frac{\Delta R}{R} \approx -\frac{\hbar}{e^2 L} R \sqrt{L_\varphi{}^2 + \left(\frac{A}{l_B}\right)^2}$$

$$L_\varphi = \sqrt{D\tau_\varphi} \qquad l_B = \sqrt{\hbar / 2eB}$$

L is the length of the wire
A is the wire cross-section

Dephasing rate can be measured

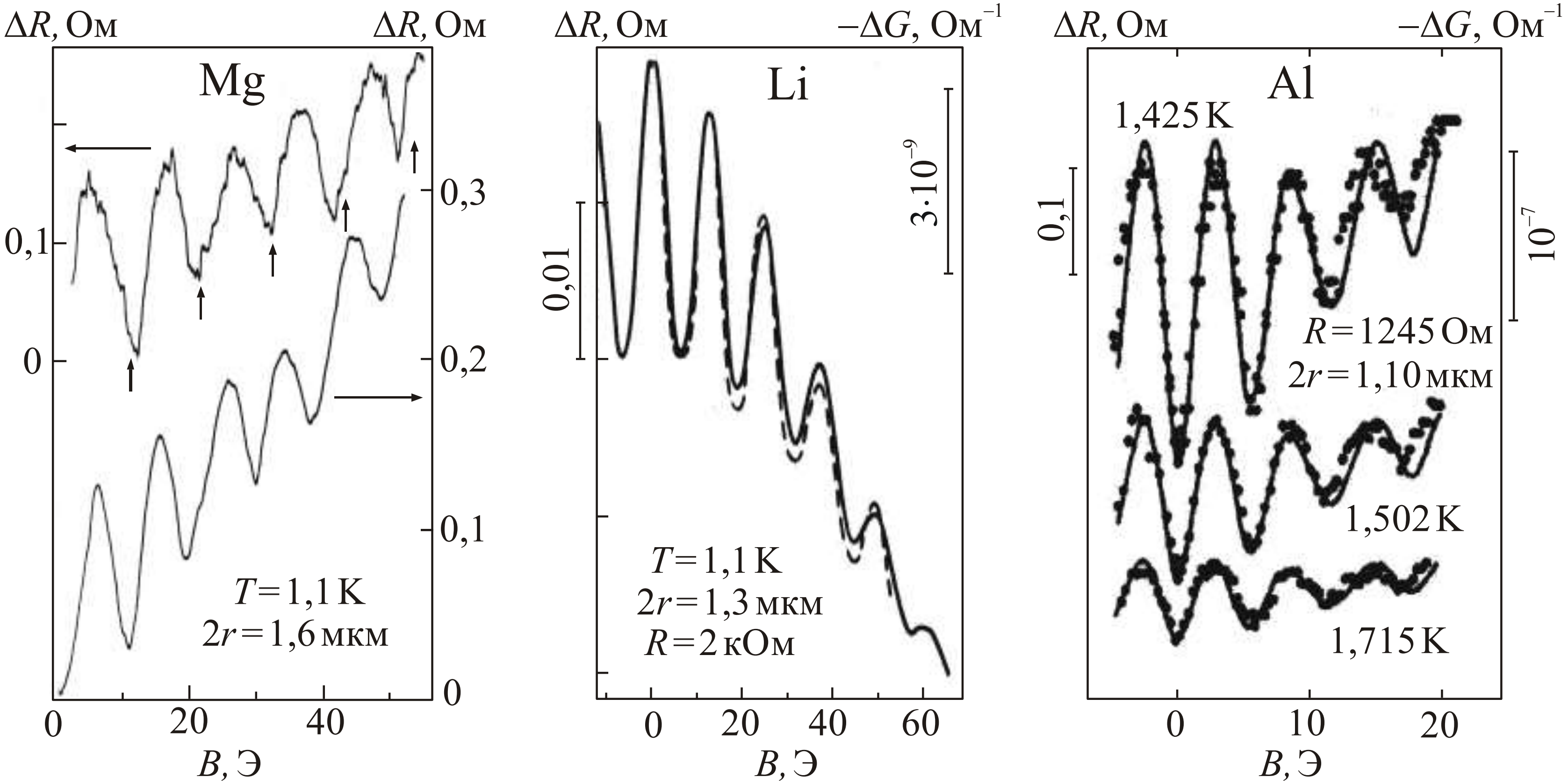
Temperature dependence of τ_{φ} (from magnetoresistance)



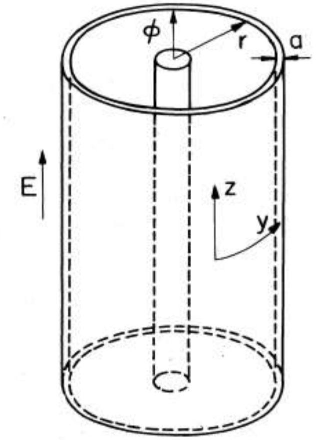
Echternach, Gershenson, Bozler, Bogdanov & Nilsson,

PRB 48, 11516 (1993)

Magnetoresistance of cylindrical films

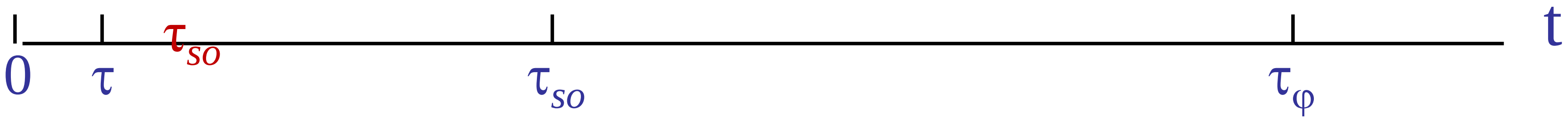


Sharvin et al, JETP Lett.
34, 285 (1981);
35, 476 (1982)



M. Gijs *et al.*,
Phys.Rev.B 30, 2964
(1984)

Weak antilocalization: spin-orbit coupling time



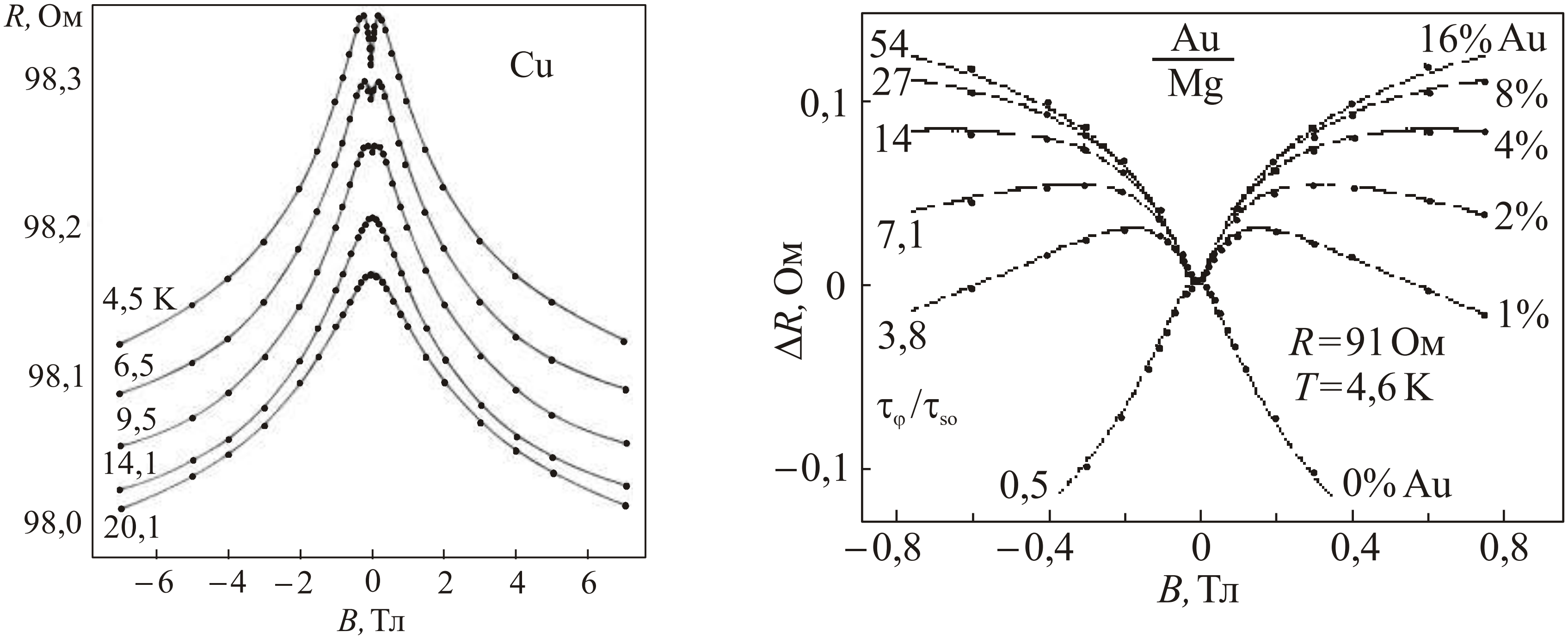
$$\Psi = \begin{pmatrix} \Psi_0 \\ \Psi_{1,-1} \\ \Psi_{1,0} \\ \Psi_{1,1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\varphi_+^{(1)}\varphi_-^{(2)} - \varphi_-^{(1)}\varphi_+^{(2)}\right) \\ \varphi_-^{(1)}\varphi_-^{(2)} \\ \frac{1}{\sqrt{2}}\left(\varphi_+^{(1)}\varphi_-^{(2)} + \varphi_-^{(1)}\varphi_+^{(2)}\right) \\ \varphi_+^{(1)}\varphi_+^{(2)} \end{pmatrix}$$

1 singlet state

3 triplet states

$$\frac{\Delta\sigma_d}{\sigma} \approx -\int_{\tau}^{\tau_\varphi} \frac{v_F \lambda^2 dt}{(Dt)^{d/2} b^{3-d}} \left(\frac{3}{2} e^{-t/\tau_{so}} - \frac{1}{2} \right), \qquad d = 1, 2, 3$$

https://recordings.reu1.blindsidenetworks.com/utwente/7e50182798188991c7a1e5d3c90def



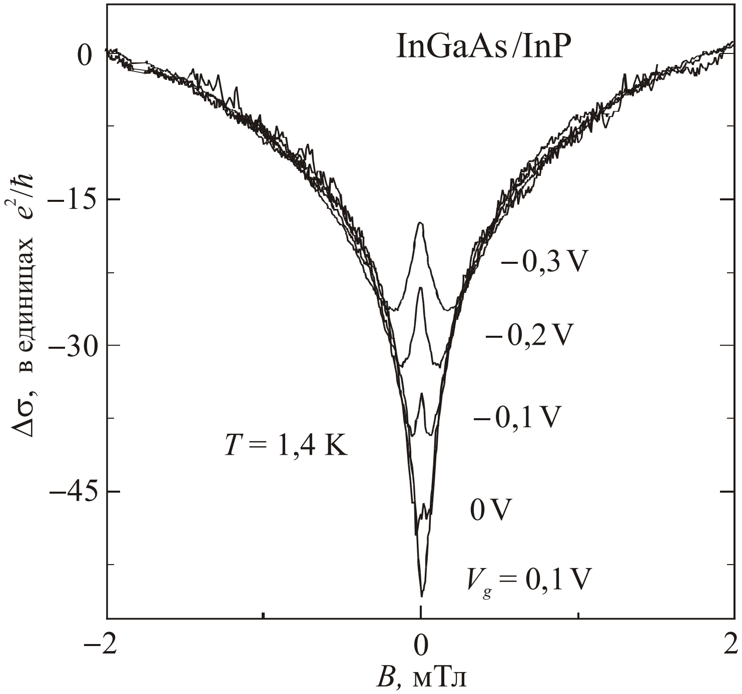
G.Bergmann, Phys.Rep. **107,** 1 (1981)

Heterostructures:

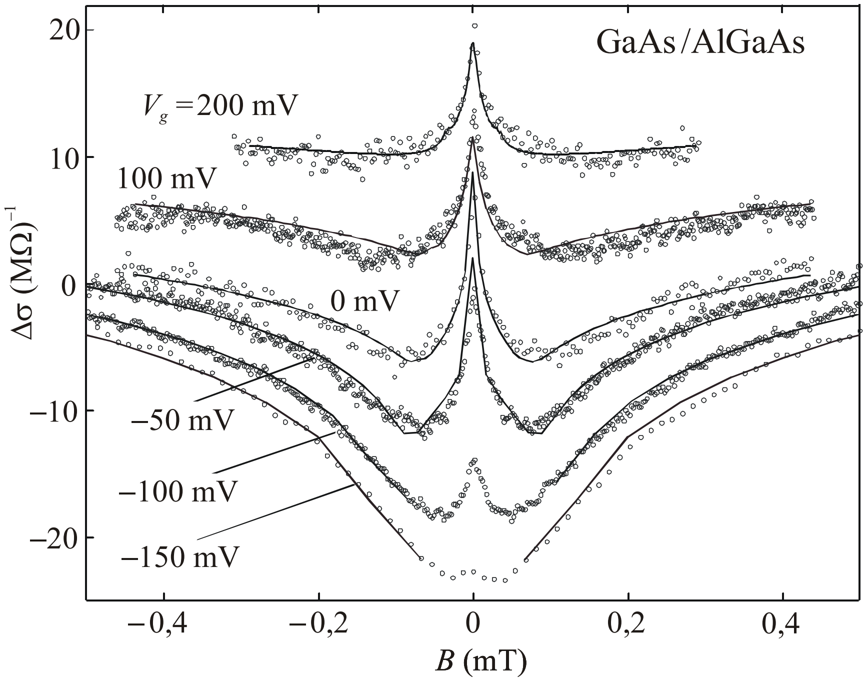
spin-orbit interaction ~ $\boldsymbol{E}\boldsymbol{e} \propto \boldsymbol{E}\,[\mu v]$

dependence on external field : via $\boldsymbol{v}$.

=> *Dependence on gate potential*

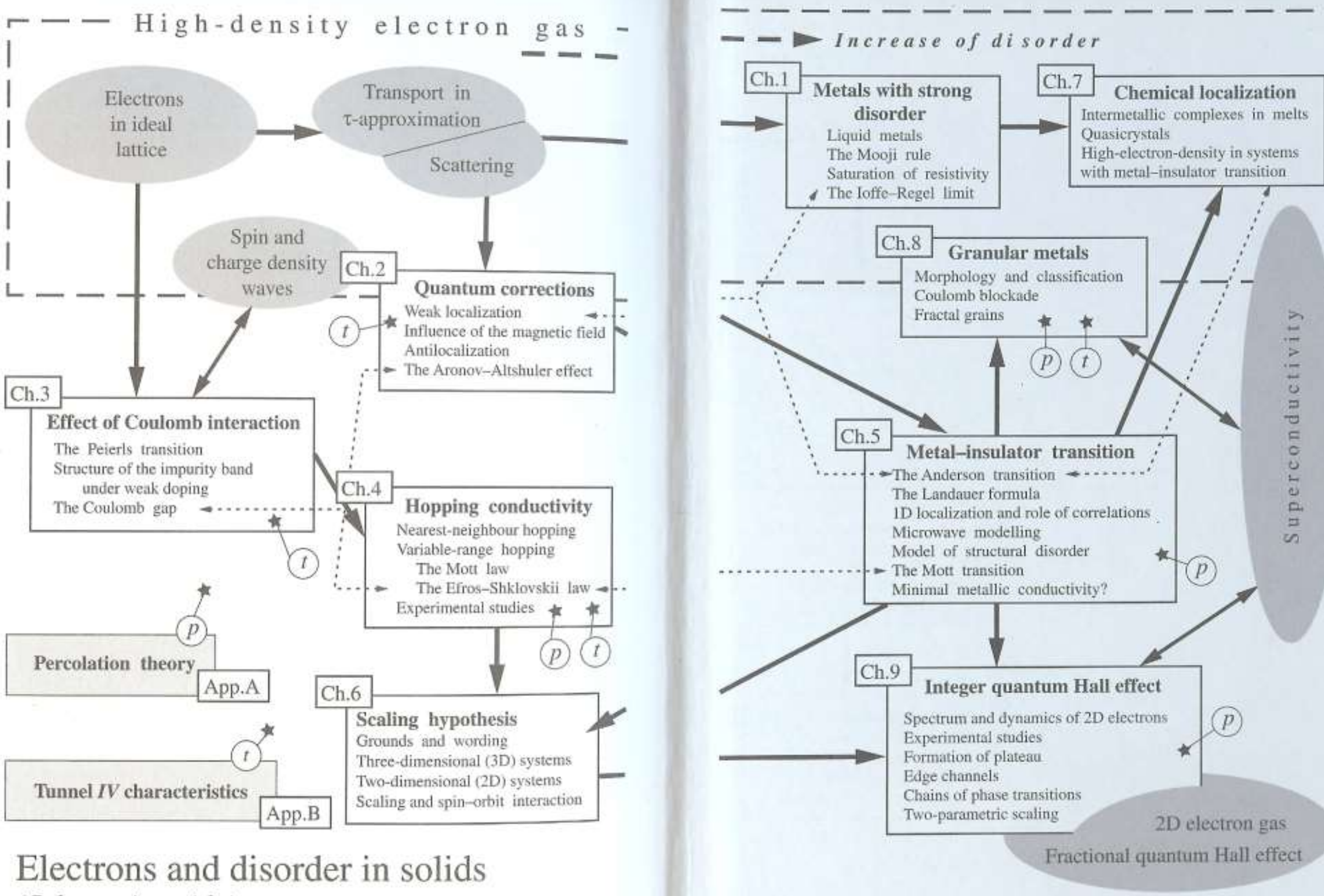


S.A. Studenikin *et al.*,
JETP L:ett. **77,** 362 (2003)



J.B. Miller *et al.*,
PRL **90,** 076807 (2003)

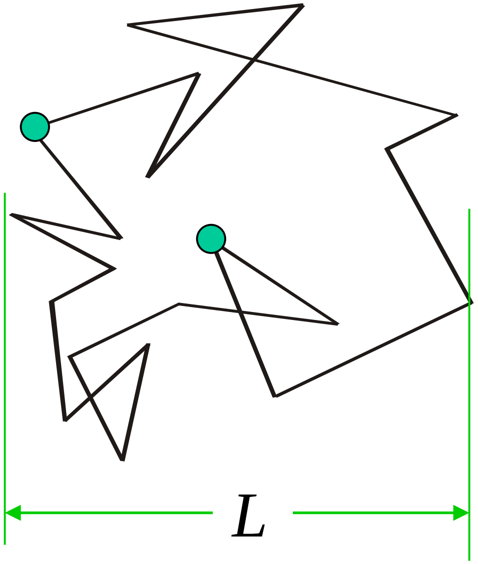
Quantum corrections to conductance due to e-e interaction (the density of states effect)



Electrons and disorder in solids

(Schematic guide)

Dephasing due to *ee* – interaction



ballistic $r \sim v_F t$

diffusive $r \sim l\sqrt{t/\tau} \sim v_F\sqrt{t\tau}$

phase $\exp(i\varphi) = \exp[i(\varepsilon_i/\hbar)t], \quad \Delta\varphi = (\Delta\varepsilon/\hbar)t$

dephasing time

$$\left.\begin{array}{r} \tau_{ee} \sim \hbar/\Delta\varepsilon \\ \Delta\varepsilon \sim T \end{array}\right\} \quad \boxed{\tau_{ee} \sim \hbar/T}$$

dephasing length

$$\boxed{L_{ee} \approx \sqrt{D\tau_{ee}} \approx \sqrt{\frac{\hbar D}{T}}}$$

Electrons diffuse and keep their coherence during time τ_{ee}

typical distance during this time is L_{ee} .

Thouless energy

Using the expression for the dephasing length

$$\boxed{L_{ee} \approx \sqrt{D\tau_{ee}} \approx \sqrt{\frac{\hbar D}{T}}}$$

and replacing L_{ee} by the system size L

we define the corresponding energy scale, ***the Thouless energy***

$$\boxed{E_T = \hbar D / L^2}$$

E_T has a meaning of the inverse diffusion time of the traveling through the system.

It determines the "phase coherent" energy interval around Fermi energy for a given system size L

Exchange interaction

Consider two electrons in states 1,2 with orbital wave functions $\varphi_1(\mathbf{r})$ and $\varphi_2(\mathbf{r})$

Total wave function $$\varphi = \frac{1}{\sqrt{2}}\left[\varphi_1(\mathbf{r}_1)\varphi_2(\mathbf{r}_2) \pm \varphi_1(\mathbf{r}_2)\varphi_2(\mathbf{r}_1)\right]$$

sign '+' for total spin S = 0, sign '-' for spin S = 1

since the sum of the orbital (L) and spin (S) quantum numbers should be even

Average values of interaction energy $U(\mathbf{r}_2 - \mathbf{r}_1)$ are $A \pm J$

$$A = \iint U\,|\varphi_1(\mathbf{r}_1)|^2\,|\varphi_2(\mathbf{r}_2)|^2\,dV_1 dV_2,$$

$$J = \iint U\varphi_1(\mathbf{r}_1)\,\varphi_1^*(\mathbf{r}_2)\,\varphi_2(\mathbf{r}_2)\,\varphi_2^*(\mathbf{r}_1)\,dV_1 dV_2$$

⇒ shifts of the energy levels $\Delta E_0 = J, \quad \Delta E_1 = -J$

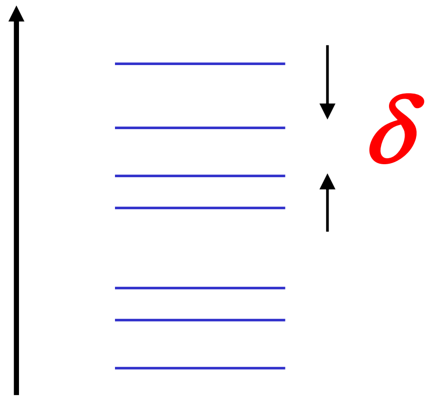
J - **exchange energy**

Exchange interaction and the density of states

Mean level spacing

$\delta = 1/gL^d$

g is the density of states



Exchange interaction leads to the shift of the energy levels $\Delta E_0 = J, \quad \Delta E_1 = -J$

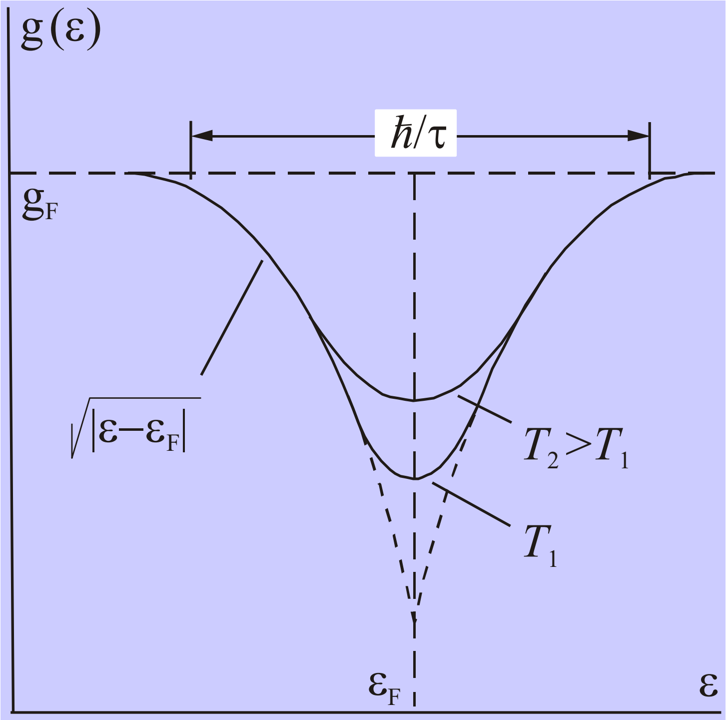
⇒ Effective "level repulsion" => Reduction of the density of states

This effect is realized only in the "phase coherent" energy range determined by

$$\tau_{ee} \sim \hbar / \Delta\varepsilon$$

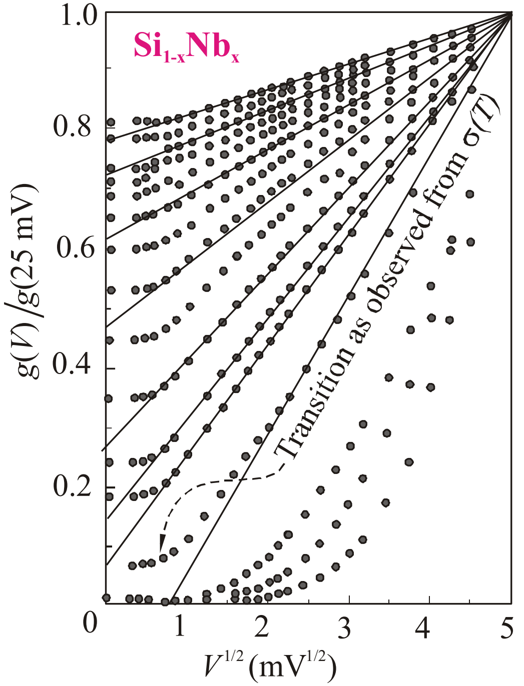
$$\Delta\varepsilon \sim T$$

The result of level repulsion - correction to the density of states: Altshuler - Aronov (AL) effect

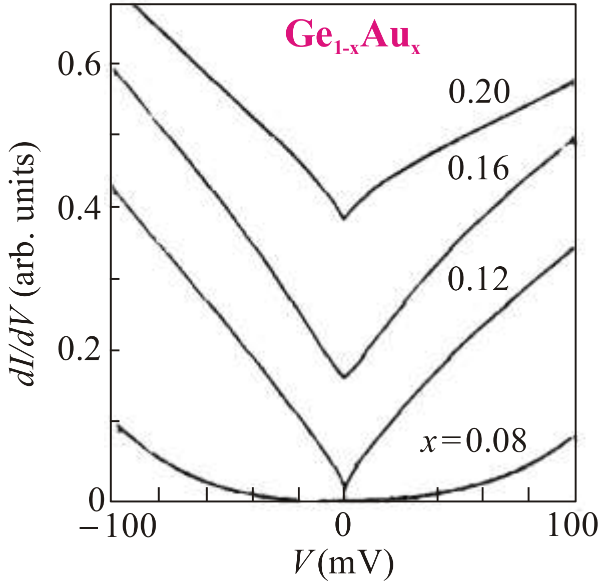


$$g(T,\varepsilon) \simeq \begin{cases} g_{\mathrm{F}}, & |\varepsilon_{\mathrm{F}} - \varepsilon| > \hbar/\tau, \\ g(0,\varepsilon), & T < |\varepsilon_{\mathrm{F}} - \varepsilon| < \hbar/\tau, \\ g(0,\varepsilon = T), & |\varepsilon_{\mathrm{F}} - \varepsilon| < T. \end{cases}$$

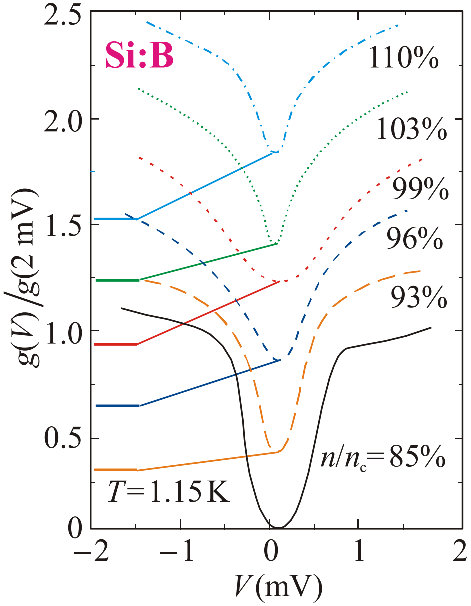
Tunneling experiments



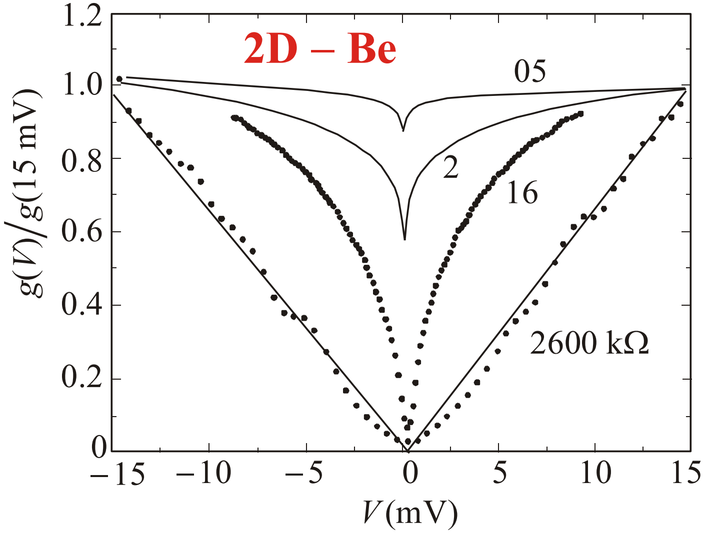
W.L. McMillan, J. Mochel
PRL **46,** 556 (1981)



G. Hertel et al.,
PRL **50,** 743 (1983)



J.G. Massey, M. Lee
PRL **77,** 3399 (1996)



V.Yu. Butko, J.F. DiTisa, P.V. Adams,
PRL **84,** 1543 (2000)

Quantum correction to the conductivity due to e-e interaction

e-e interaction influencrs transport via correction to the density of states. The correction is proportional to the probability of e-e collision within time

$$\boxed{\tau_{ee} \sim \hbar/T}$$

$$\int_{\tau}^{\hbar/T} \frac{v_F \lambda^2 dt}{(Dt)^{d/2} b^{3-d}}, \qquad d = 1, 2, 3$$

=> Altshuller – Aronov effect:

$$\Delta_{ee}\sigma_3 \approx \frac{e^2}{\hbar}\left(\frac{1}{L_{ee}} - \frac{1}{l}\right)$$

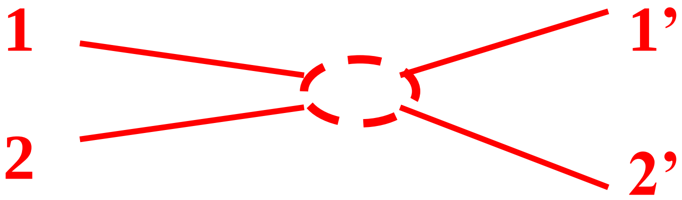
$$\Delta_{ee}\sigma_2 \approx -\frac{e^2}{\hbar}\ln \frac{L_{ee}}{l}$$

$$\Delta_{ee}\sigma_1 \approx \frac{e^2}{\hbar}(l - L_{ee})$$

$$\boxed{L_{ee} \approx \sqrt{D\tau_{ee}} \approx \sqrt{\frac{\hbar D}{T}}}$$

Influence of diffusion on the frequency of ee-collisions

ballistic regime



$$\hbar/\tau_e \sim T^2/\varepsilon_F$$

diffusive regime

Size of interaction region $L_{ee} >> 1/k_F$, *momentum transfer is small* **:**

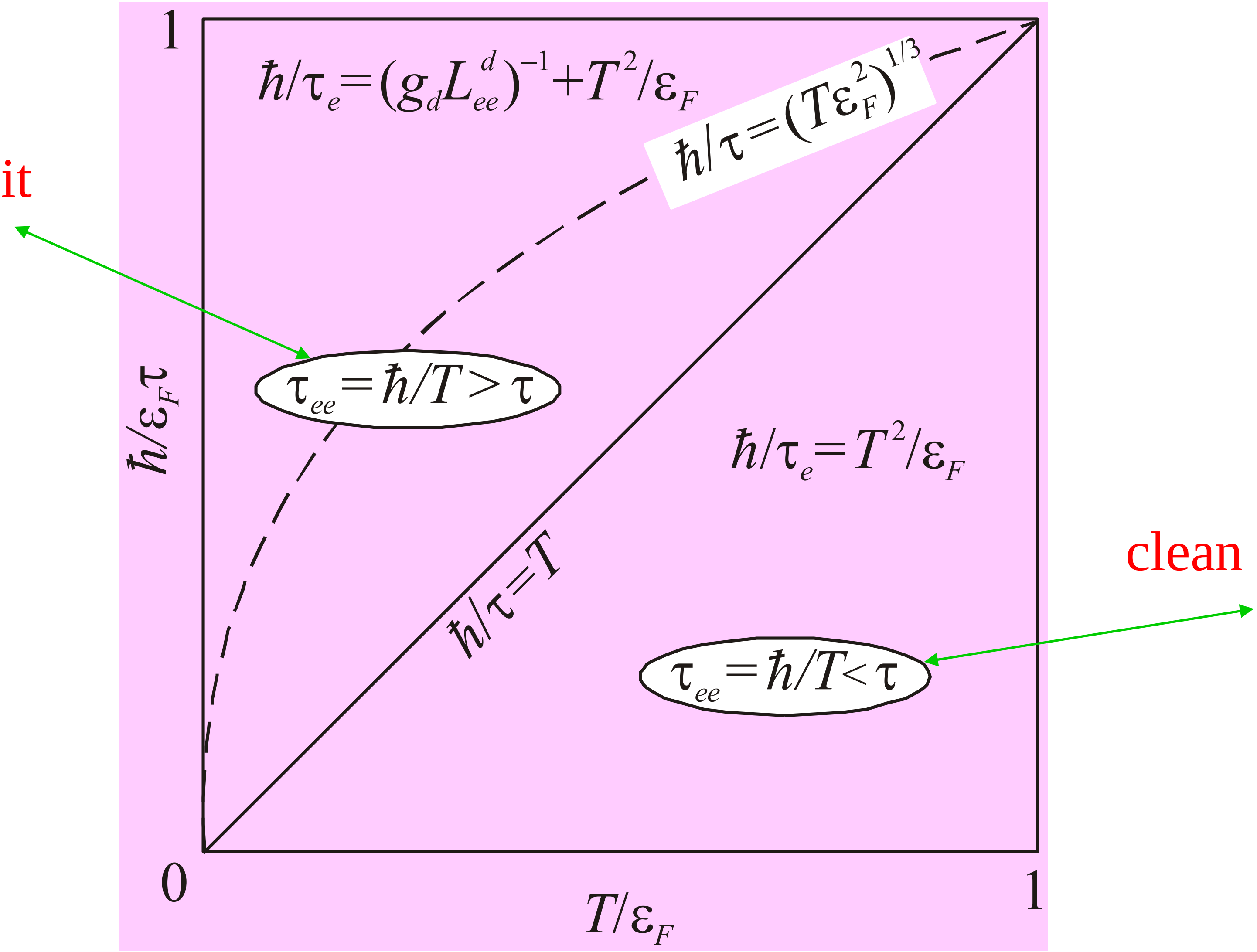
$$q \approx 1/L_{ee} << k_F$$

$$\hbar/\tau_e \sim q^d/g_d \sim 1/g_d L_{ee}^d$$

, *density of states* $g_d \sim \varepsilon_F^{d/2-1} m^{1/2}$

$$\frac{\hbar}{\tau_e} \sim T^{d/2}\varepsilon_F^{1-d}\tau^{-d/2} = \begin{cases} T^{1/2}\tau^{-1/2}, & d=1 \\ T\varepsilon_F^{-1}\tau^{-1}, & d=2 \\ T^{3/2}\varepsilon_F^{-2}\tau^{-3/2}, & d=3 \end{cases}$$

This time constant τ_e controls the weak localization processes

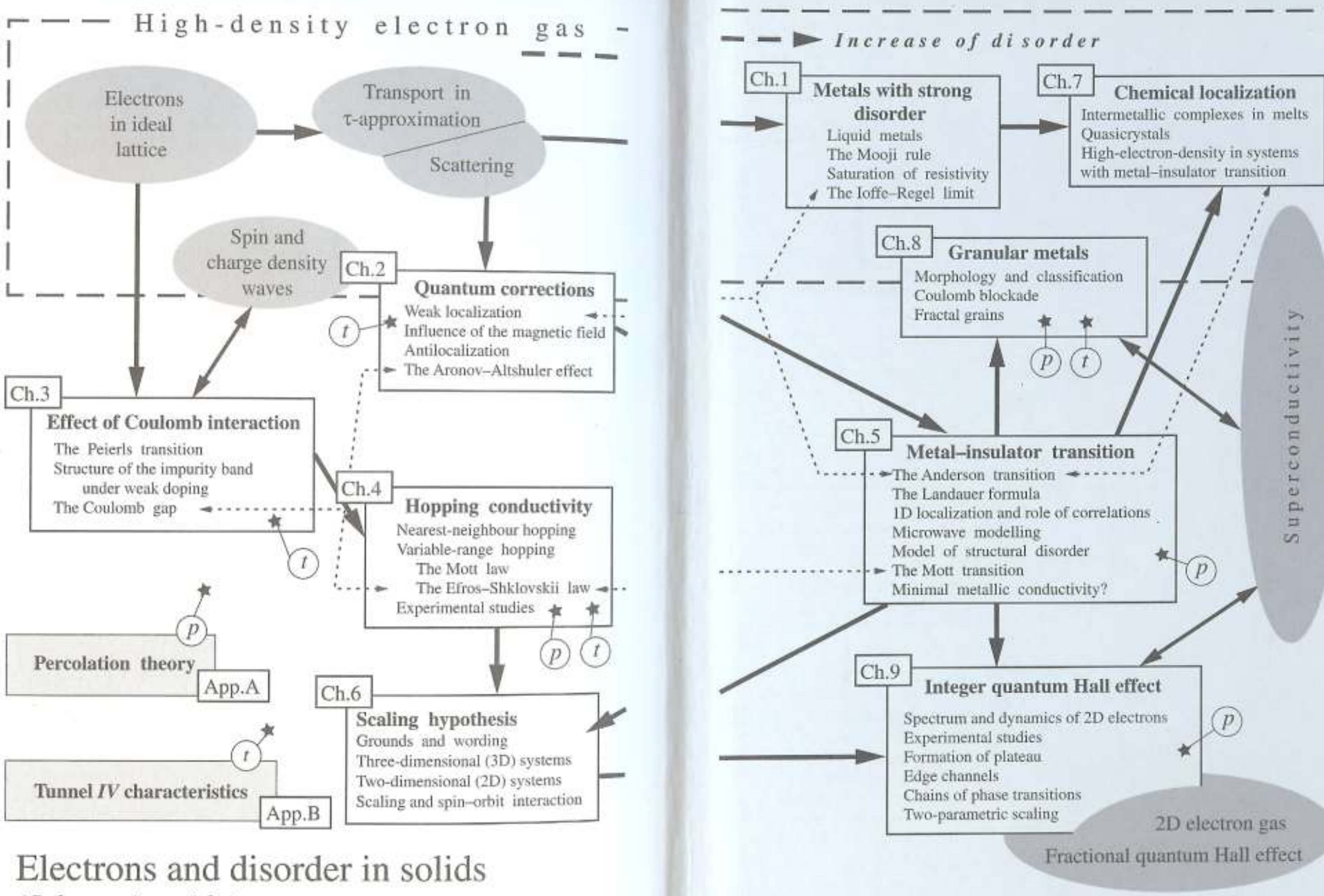


dirty limit

clean limit

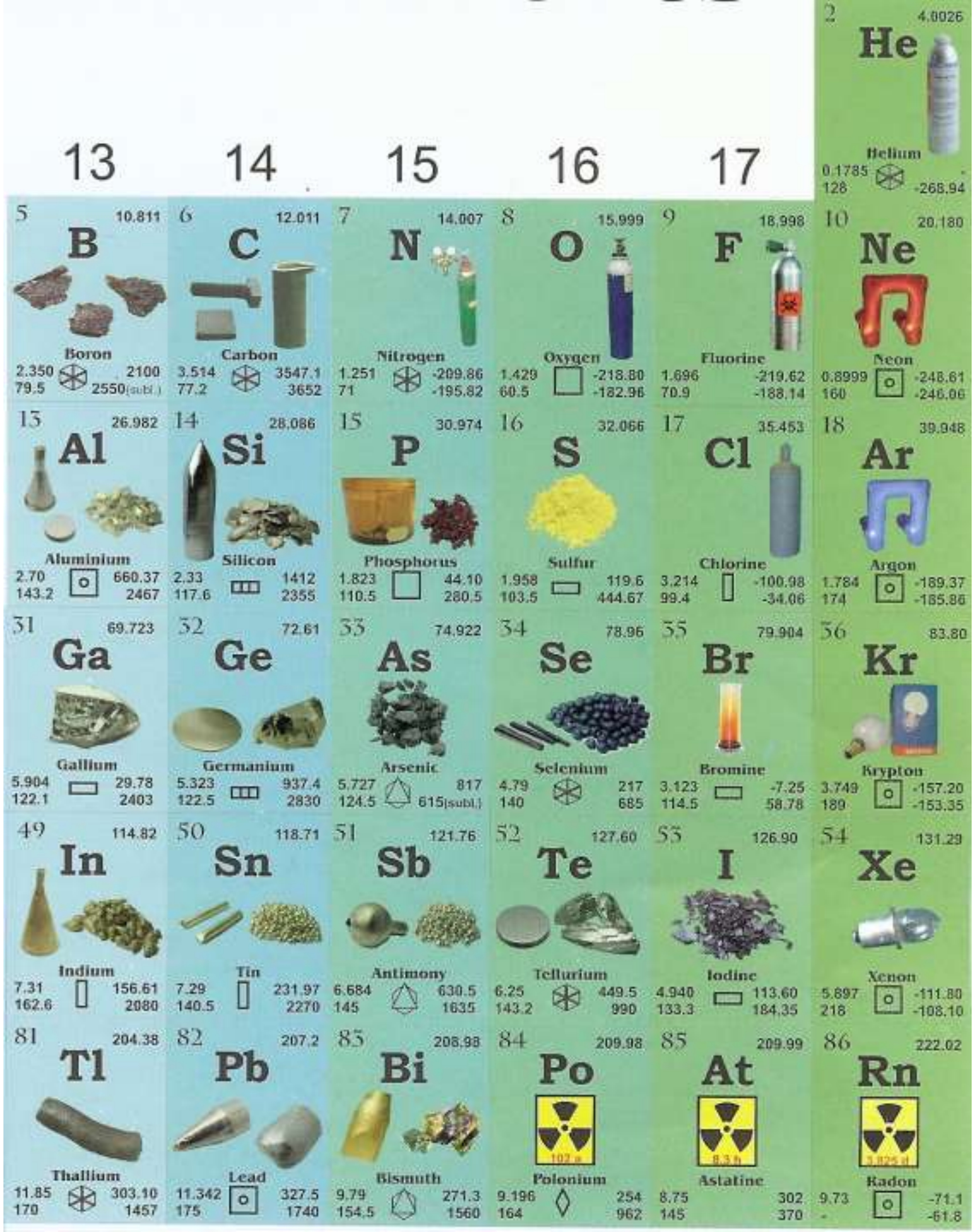
$$\frac{1}{\tau_{eff}}=\frac{1}{\tau}+\frac{1}{\tau_{ph}}$$

Electronic Phase Transitions



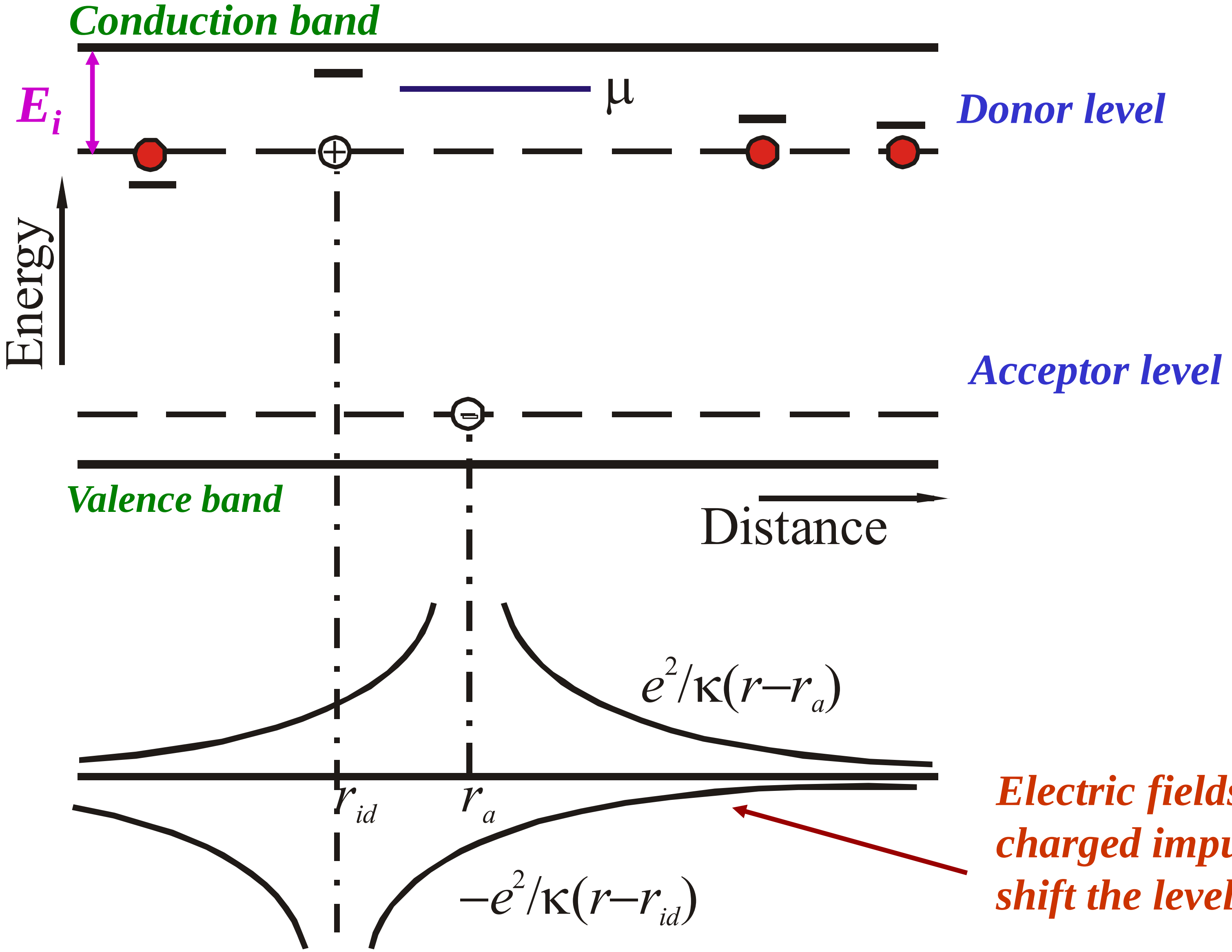
Electrons and disorder in solids

(Schematic guide)



13	14	15	16	17	
					2 4.0026 **He** Helium 0.1785 - 128 -268.94
5 10.811 **B** Boron 2.350 2100 79.5 2550(subl.)	6 12.011 **C** Carbon 3.514 3547.1 77.2 3652	7 14.007 **N** Nitrogen 1.251 -209.86 71 -195.82	8 15.999 **O** Oxygen 1.429 -218.80 60.5 -182.96	9 18.998 **F** Fluorine 1.696 -219.62 70.9 -188.14	10 20.180 **Ne** Neon 0.8999 -248.61 160 -246.06
13 26.982 **Al** Aluminium 2.70 660.37 143.2 2467	14 28.086 **Si** Silicon 2.33 1412 117.6 2355	15 30.974 **P** Phosphorus 1.823 44.10 110.5 280.5	16 32.066 **S** Sulfur 1.958 119.6 103.5 444.67	17 35.453 **Cl** Chlorine 3.214 -100.98 99.4 -34.06	18 39.948 **Ar** Argon 1.784 -189.37 174 -185.86
31 69.723 **Ga** Gallium 5.904 29.78 122.1 2403	32 72.61 **Ge** Germanium 5.323 937.4 122.5 2830	33 74.922 **As** Arsenic 5.727 817 124.5 615(subl.)	34 78.96 **Se** Selenium 4.79 217 140 685	35 79.904 **Br** Bromine 3.123 -7.25 114.5 58.78	36 83.80 **Kr** Krypton 3.749 -157.20 189 -153.35
49 114.82 **In** Indium 7.31 156.61 162.6 2080	50 118.71 **Sn** Tin 7.29 231.97 140.5 2270	51 121.76 **Sb** Antimony 6.684 630.5 145 1635	52 127.60 **Te** Tellurium 6.25 449.5 143.2 990	53 126.90 **I** Iodine 4.940 113.60 133.3 184.35	54 131.29 **Xe** Xenon 5.897 -111.80 218 -108.10
81 204.38 **Tl** Thallium 11.85 303.10 170 1457	82 207.2 **Pb** Lead 11.342 327.5 175 1740	83 208.98 **Bi** Bismuth 9.79 271.3 154.5 1560	84 209.98 **Po** 102 a Polonium 9.196 254 164 962	85 209.99 **At** 8.3 h Astatine 8.75 302 145 370	86 222.02 **Rn** 3.825 d Radon 9.73 -71.1 - -61.8

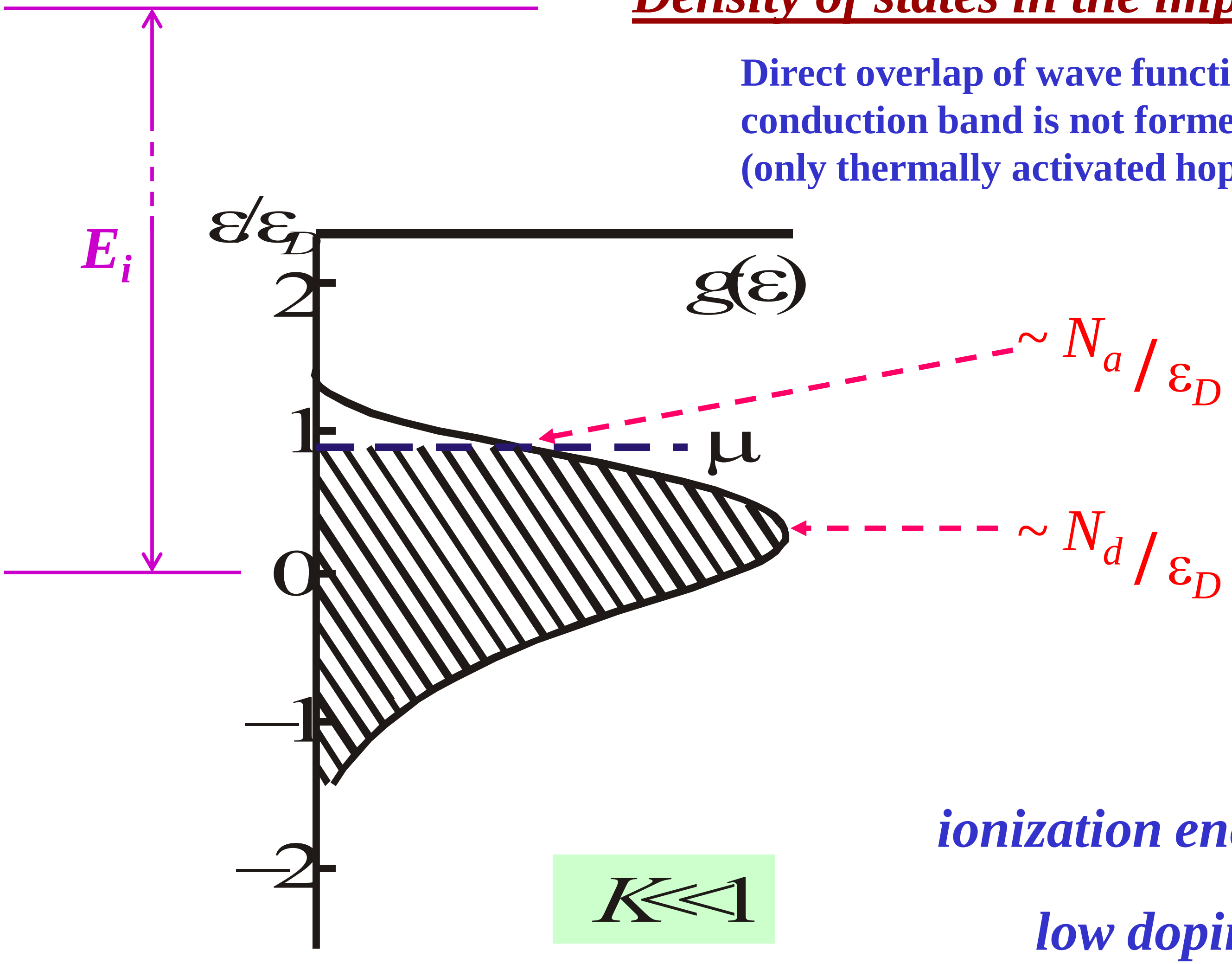
Impurity band



Electric fields of charged impurities shift the levels

Density of states in the impurity band

Direct overlap of wave functions is negligible
conduction band is not formed => insulator at T=0
(only thermally activated hopping is possible)



Bohr radius

$$a_B = \frac{\kappa\hbar^2}{m^* e^2}$$

ionization energy $E_i = \dfrac{e^2}{2a_B\kappa}$

low doping:

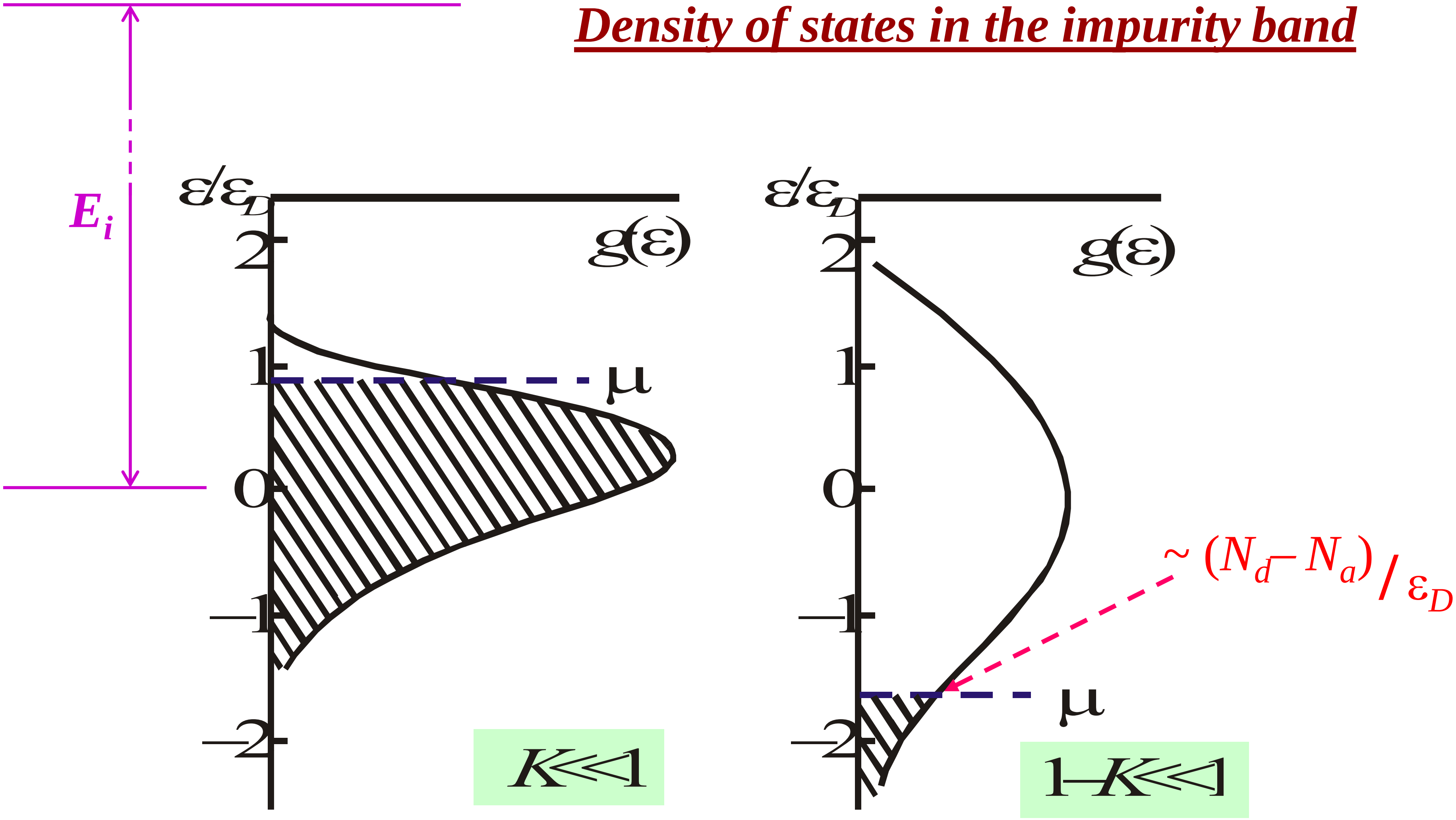
$$N a_B^3 \ll 1, \quad \Rightarrow \quad \varepsilon_D \ll E_i$$

$K \ll 1$

compensation $K = N_a / N_d$

band width $\varepsilon_D = \dfrac{e^2}{\kappa} N_d^{1/3}$

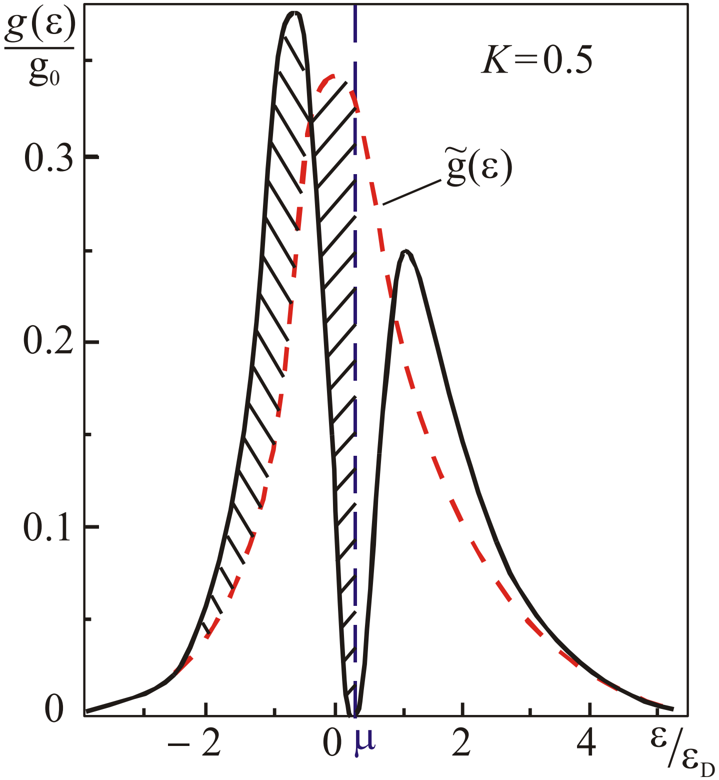
Density of states in the impurity band



Weak compensation

Strong compensation

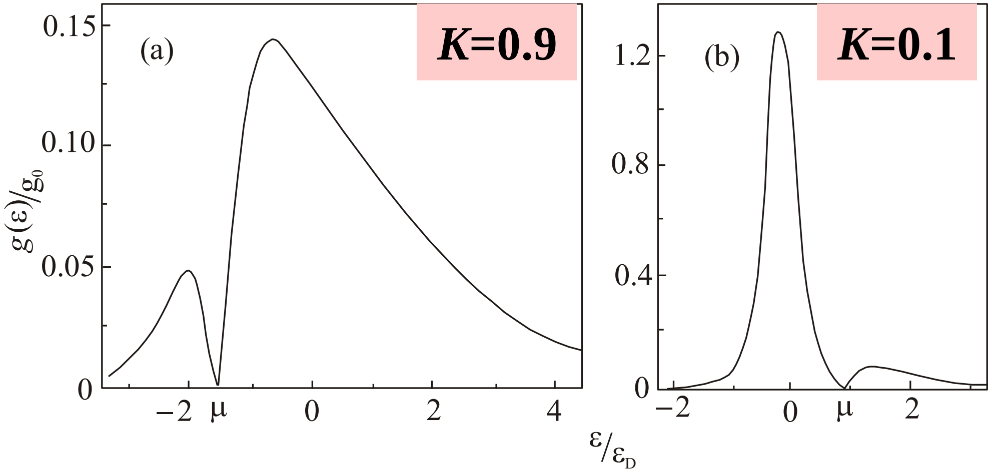
Coulomb gap



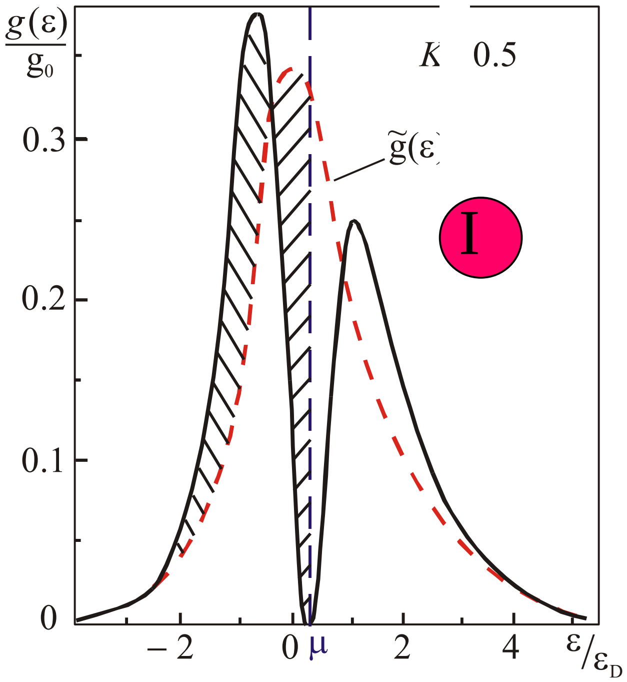
Energy of occupied states decreased

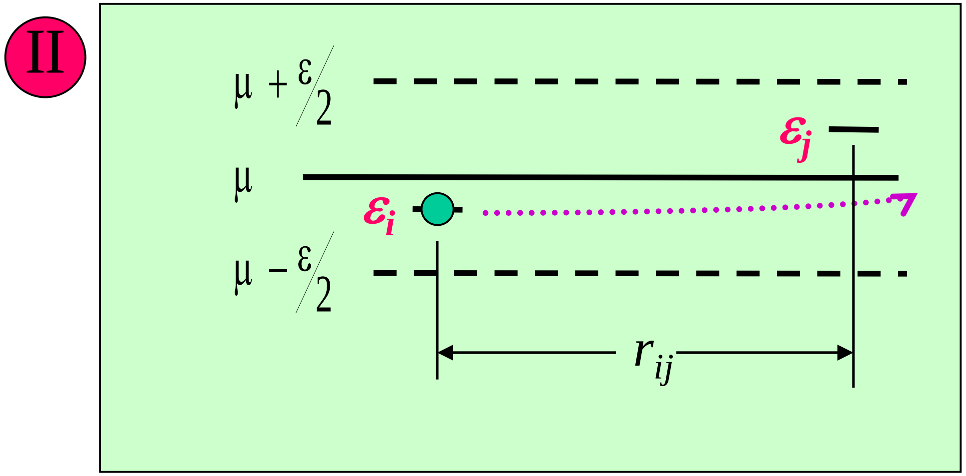
Energy of empty states increased

A.L. Efros, N.V. Lien, B.I. Shklovskii
J. Phys. C **12,** 1023 (1979)

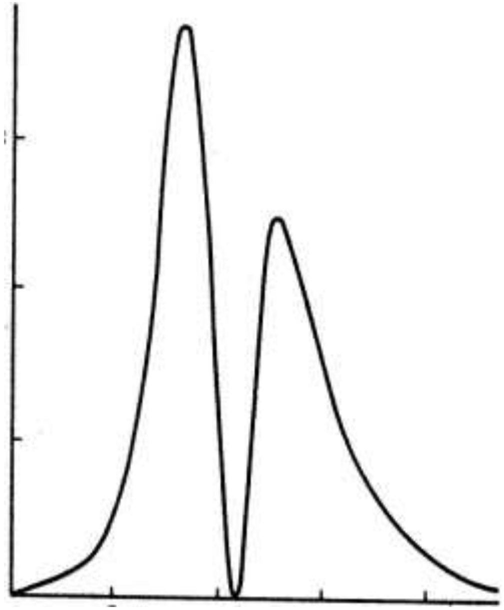


Coulomb gap





$$\varepsilon_j - \varepsilon_i - \frac{e^2}{\kappa r_{ij}} \geq 0 \longrightarrow \frac{e^2}{\kappa r_{ij}} \leq \varepsilon$$



3D

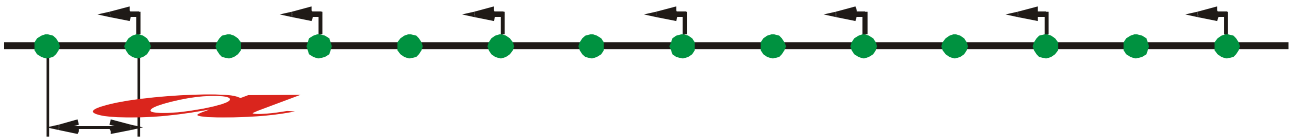
$$N(\varepsilon) = r_{ij}^{-3} \leq \left(\frac{\kappa\varepsilon}{e^2}\right)^3, \quad g(\varepsilon) = \frac{\partial N}{\partial \varepsilon} \propto \frac{\varepsilon^2 \kappa^3}{e^6}$$

2D

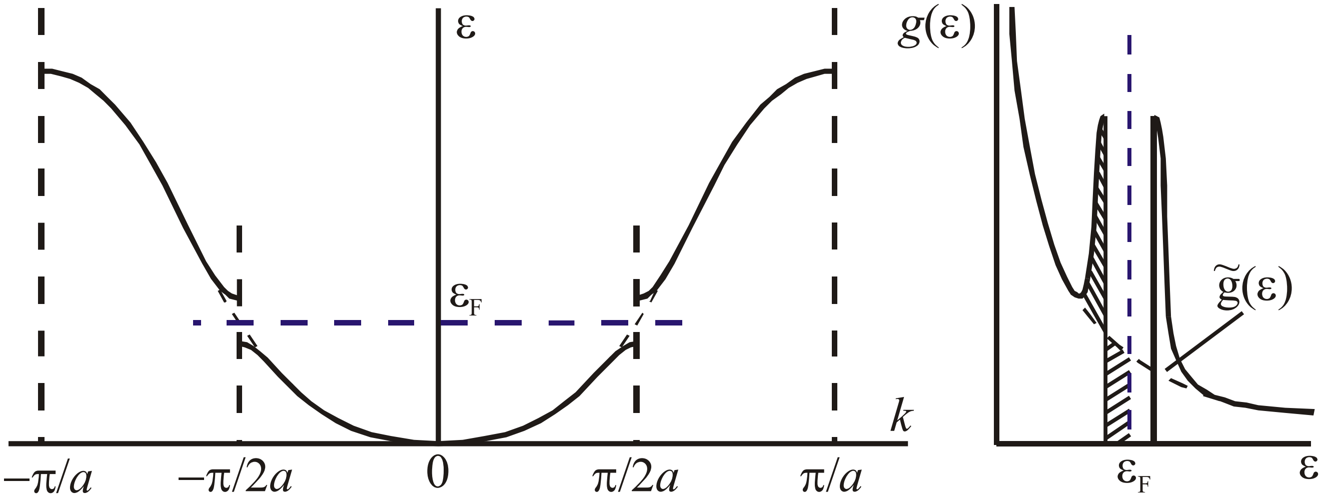
$$N(\varepsilon) = r_{ij}^{-2} \leq \left(\frac{\kappa\varepsilon}{e^2}\right)^2, \quad g(\varepsilon) = \frac{\partial N}{\partial \varepsilon} \propto \frac{|\varepsilon|\kappa^2}{e^4}$$

Another example of electronic phase transition:

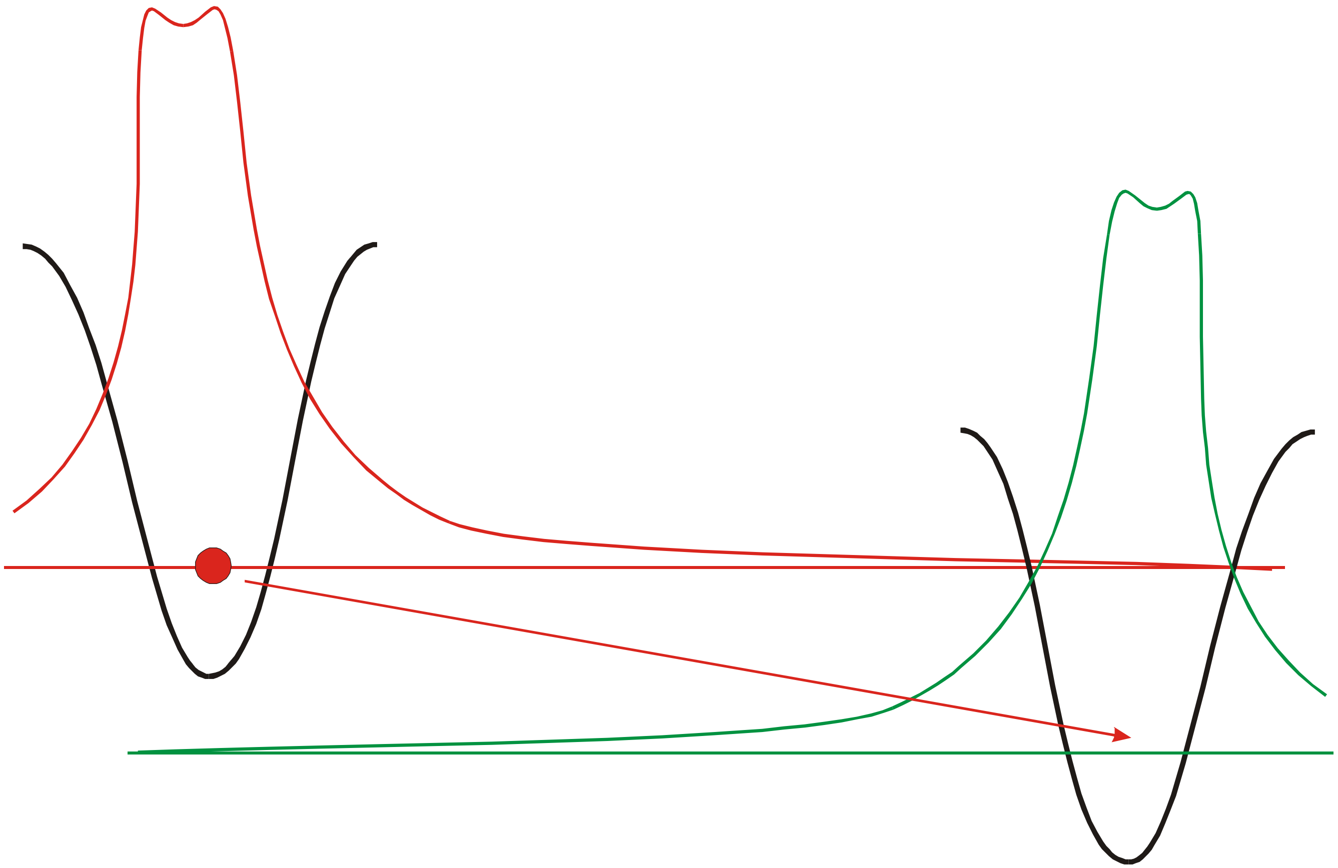
Peierls transition



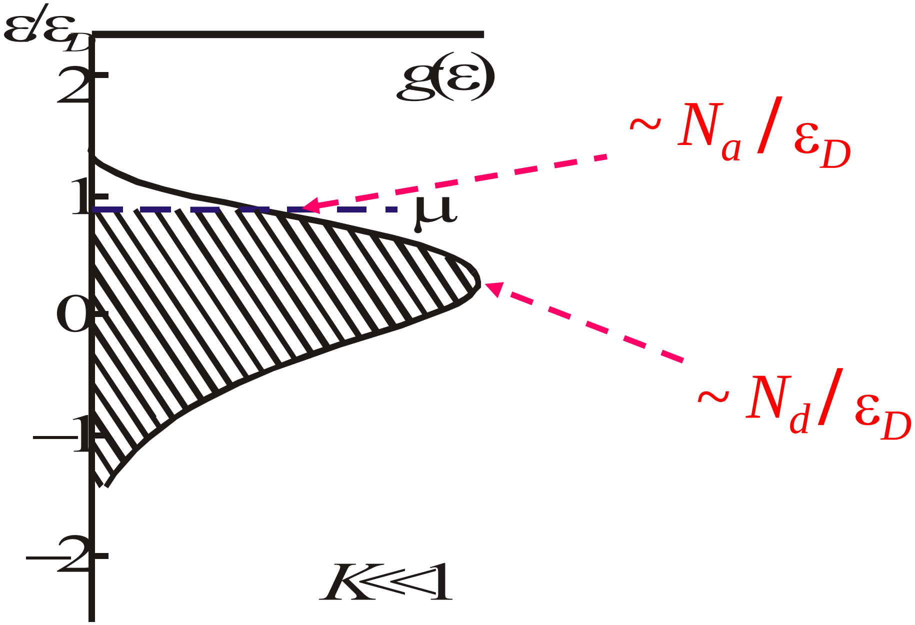
Period doubling $a \rightarrow 2a$



Hopping conductivity



Main characteristics of an impurity band



Bohr radius

$$a_B = \frac{\kappa\hbar^2}{m^* e^2}$$

band width $\varepsilon_D = \frac{e^2}{\kappa} N_d^{1/3}$

low doping regime: $a_B << r_{ij}$

where r_{ij} is the distance between neighboring sites

Compensation factor $K = N_a / N_d$

$$Na_B^3 << 1, \quad \Rightarrow \quad \varepsilon_D << E_i$$

Hydrogen atom: attractive Coulomb potential $U = -(e^2/r)$

the Bohr radius $a_{\mathrm{B}} = \hbar^2/me^2$

$$\psi_n(r) \to C(n) r^{n-1} \exp(-r/na_{\mathrm{B}}) \text{ as } r \to \infty, \quad (n = 1, 2, 3, \ldots)$$

In the ground state (n=1) localization length equals a_B

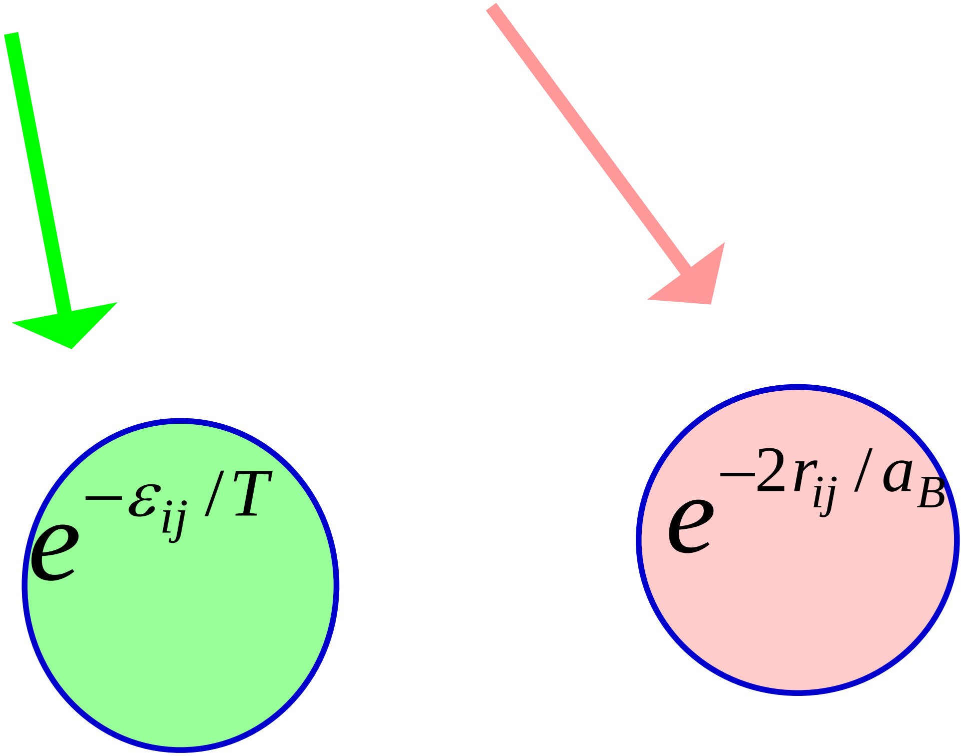
Hopping probability

$$\frac{1}{\tau_{ij}} \propto \quad F(\varphi_{ij}, f_i, f_j)\left|\int \psi_j^* e^{iqr}\psi_i d^3r\right|^2$$

$$f_i = (\exp\frac{\varepsilon_i - \mu}{T} + 1)^{-1}$$

$$\varphi_{ij} = (\exp\frac{\varepsilon_{ij}}{T} - 1)^{-1}$$

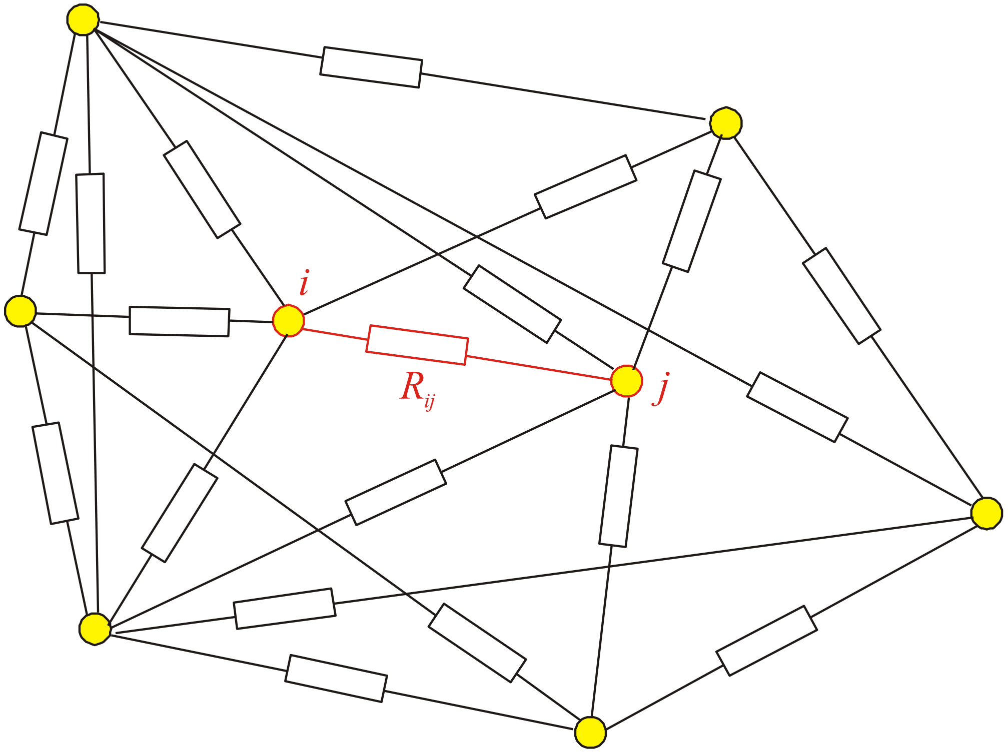
$$\varepsilon_{ij} = \varepsilon_j - \varepsilon_i$$



$$e^{-\varepsilon_{ij}/T}$$

$$e^{-2r_{ij}/a_B}$$

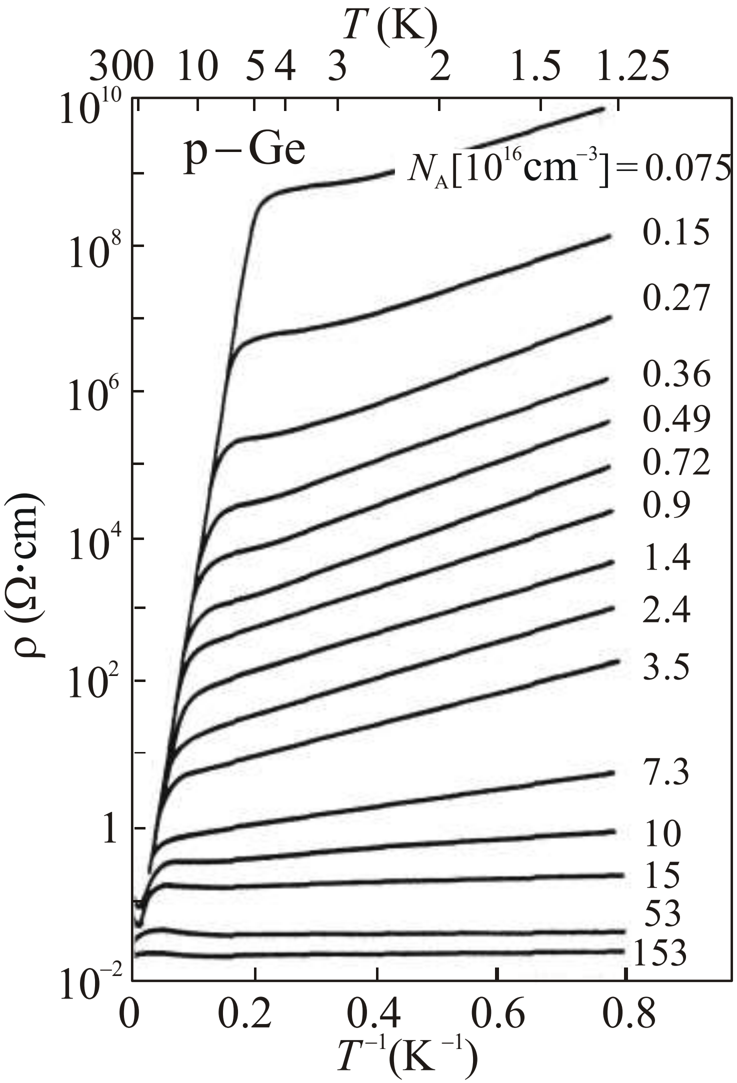
Abrahams-Miller net



$$R_{ij} = R_0 e^{u_{ij}}$$

$$u_{ij} = \frac{2r_{ij}}{a_B} + \frac{\varepsilon_{ij}}{T}$$

Experiment



H.Fritzsche, M.Guevas, PR **119,** 1238 (1960)

Neutron-irradiated Ge

As a result of nuclear reaction

one of Ge isotopes -> Ga
another Ge isotope - > As

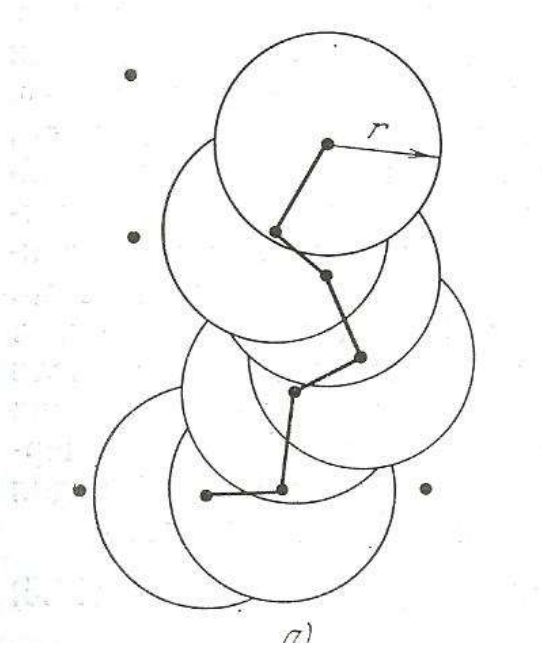
Ga –acceptors, As – donors

K = 0.4 is fixed

while N_a is a function of irradiation time

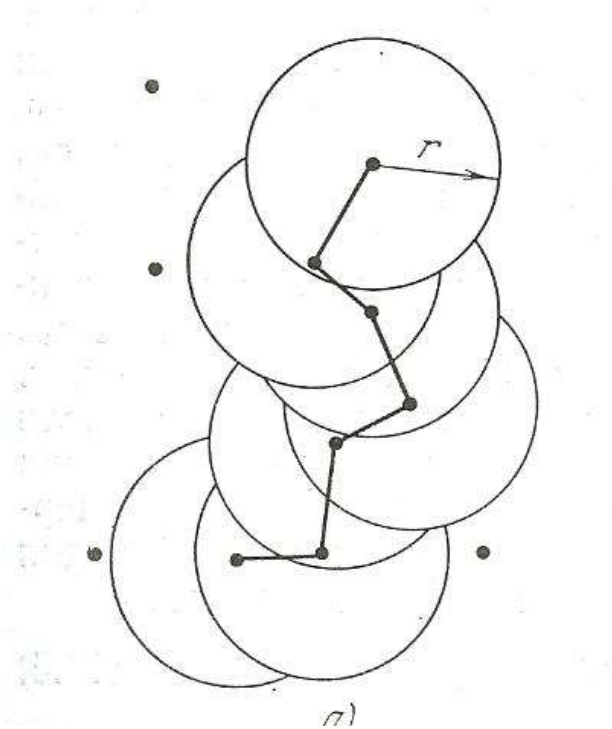
Basic ideas of percolation theory

- *The nodes are introduced which are characterized by concentration* n *and the radius* r *of interaction between neighboring nodes*



- *Percolation is the problem of global connectivity across the whole sample via connected nodes*
- *In the case of electrons the interaction radius* r *is controlled by* a_B

Percolation theory: Random nodes



The percolation threshold depends on a number of nodes within the interaction radius r

The number of nodes in a sphere with radius **r**

$$\frac{4\pi}{3} r^3 n$$

where n is the concentration of nodes

Numerical results for critical concentration n_c

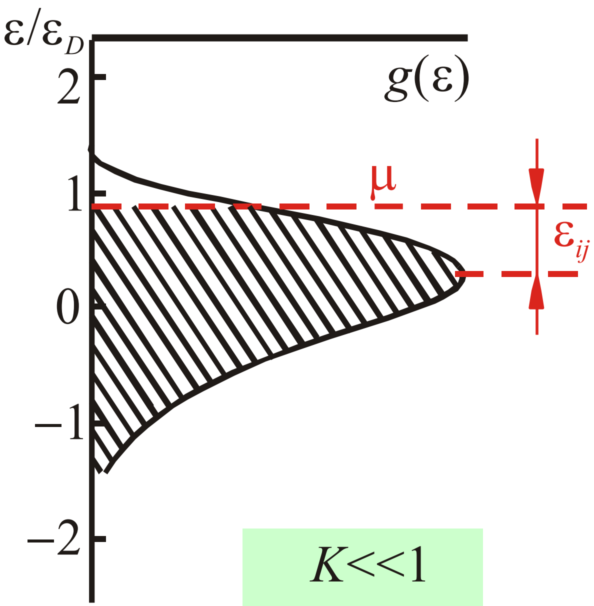
3D

$$\boxed{\frac{4\pi}{3} r_c^3 n_c = B_c^{(3)} = 2.7}$$

2D

$$\boxed{\pi r_c^2 n_c = B_c^{(2)} = 4.4}$$

Nearest neighbour hopping



$K<<1$

Abrahams-Miller net

$$R_{ij} = R_0 e^{u_{ij}}$$

$$u_{ij} = \frac{2r_{ij}}{a_B} + \frac{\varepsilon_{ij}}{T}$$

$$e^{-\varepsilon_{ij}/T}$$

T-dependent factor is the same

for all transitions

$$\frac{4\pi}{3} r_c^3 n = B_c = 2.7 \quad => \quad r_c = 0.865\, n^{-1/3}$$

Dependence on concentration: r_{ij} in the Abrahams-Miller relation is replaced by r_c =>

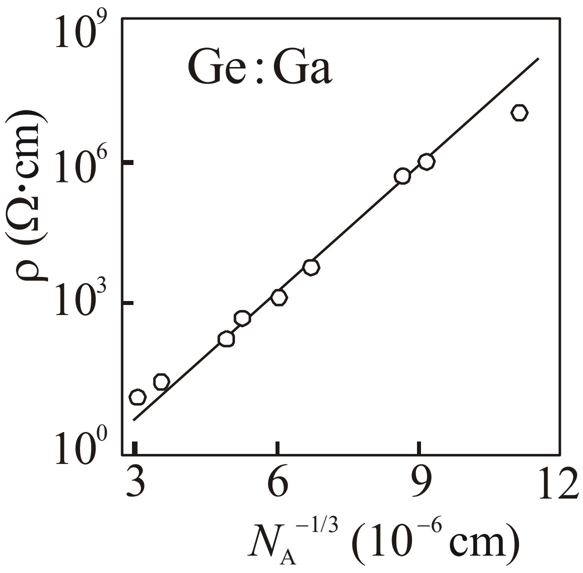
$$\boxed{\rho = \rho_0 \exp\left(1.73 \big/ n^{1/3} a_B \right)}$$

Experiment

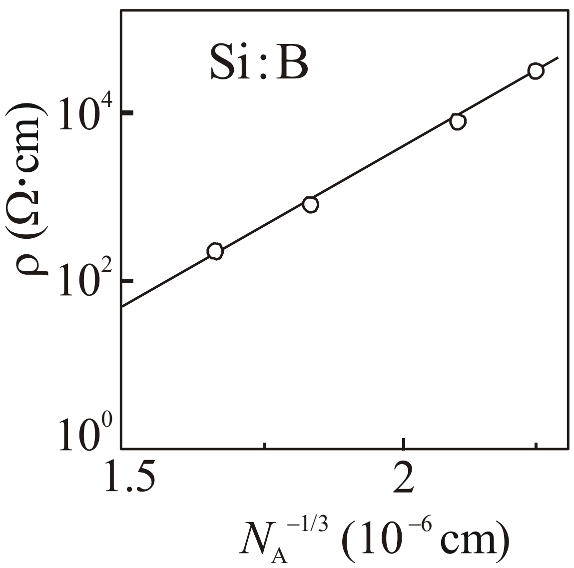
Percolation threshold $\frac{4\pi}{3} r_c^3 n = B_c = 2.7$ => $r_c = 0.865\, n^{-1/3}$

$$\boxed{\rho = \rho_0 \exp\left(1.73 \big/ n^{1/3} a_B \right)}$$

H.Fritzsche, M.Guevas,
Phys. Rev. **119,** 1238 (1960)
Ga –acceptors, As - donors

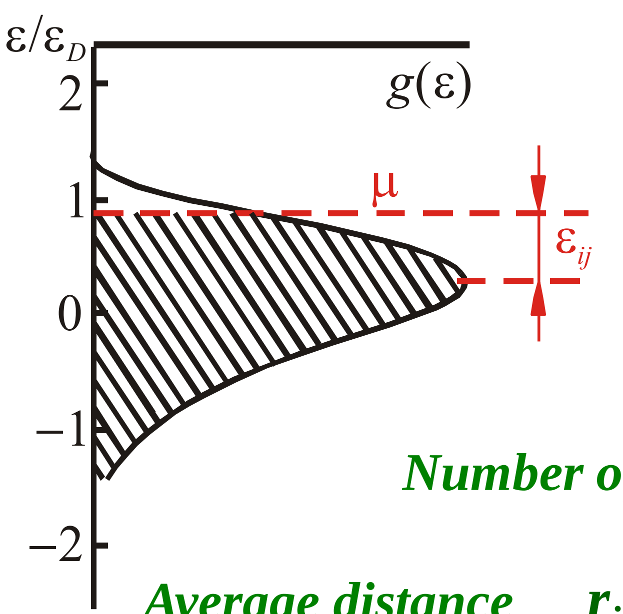


R. Ray, H.Fan,
Phys. Rev. **121,** 768 (1961)

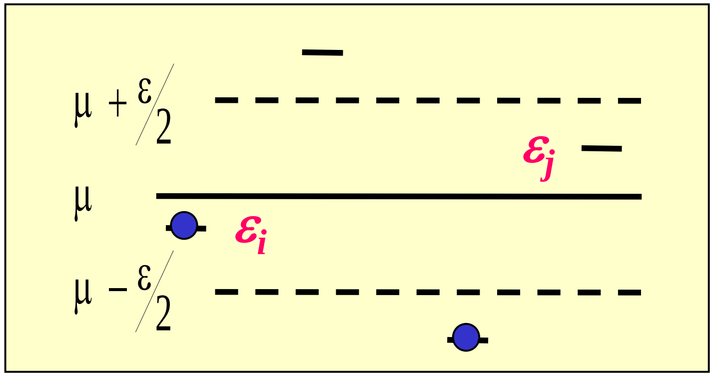


n-GaAs	1.7	1.88	1.9
n-InP	1.9		
p-Ge	1.9	1.75	2.0
p-Si	1.8		

Variable range hopping: Mott law



$T << \mu \approx \varepsilon_D$



Number of states $N(\varepsilon) = g_\mu \varepsilon$,

Average distance $r_{ij}(\varepsilon) = [N(\varepsilon)]^{-1/3}$, ***average energy spacing is*** ε

parameter u_{ij} of Miller –Abrahams net

$$u_{ij} = \frac{2}{a_B[N(\varepsilon)]^{1/3}} + \frac{\varepsilon}{T} = \frac{2}{g_\mu^{1/3} a_B \varepsilon^{1/3}} + \frac{\varepsilon}{T}$$

u_{ij} depends on ε and reaches its minimum when $\frac{d}{d\varepsilon} u_{ij}(\varepsilon) = 0$

=>

$$\varepsilon_{\min} = \left(\frac{T}{a_B g_\mu^{1/3}} \right)^{3/4} = (T^3 T_{Mott})^{1/4}, \quad T_{Mott} = (a_B^3 g_\mu)^{-1}$$

Mott law

Average hopping length $\bar{r} = r_{ij}(\varepsilon_{\min})$

$$\bar{r} = (g_\mu \varepsilon_{\min})^{-1/3} = a_B \left(T_{Mott}/T \right)^{1/4}$$

Resistance

$$\boxed{\rho = \rho_0 \exp\left(T_{Mott}/T \right)^{1/4} \qquad (d = 3)}$$

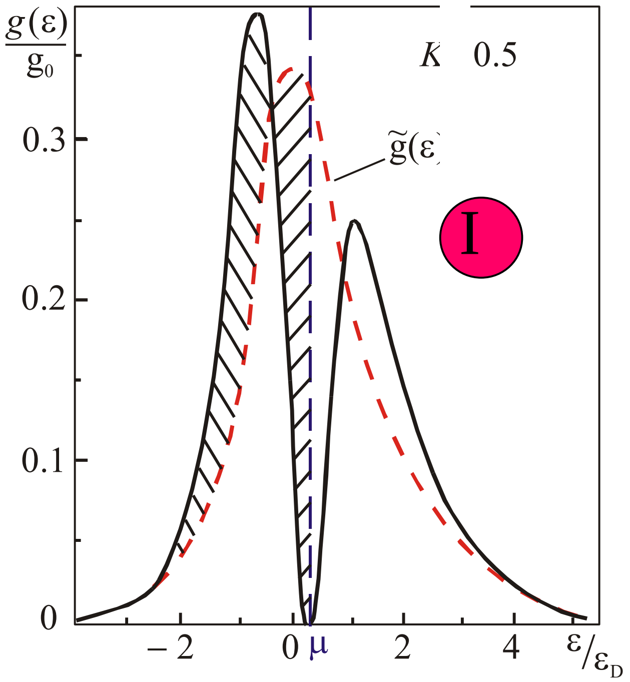
For d=2

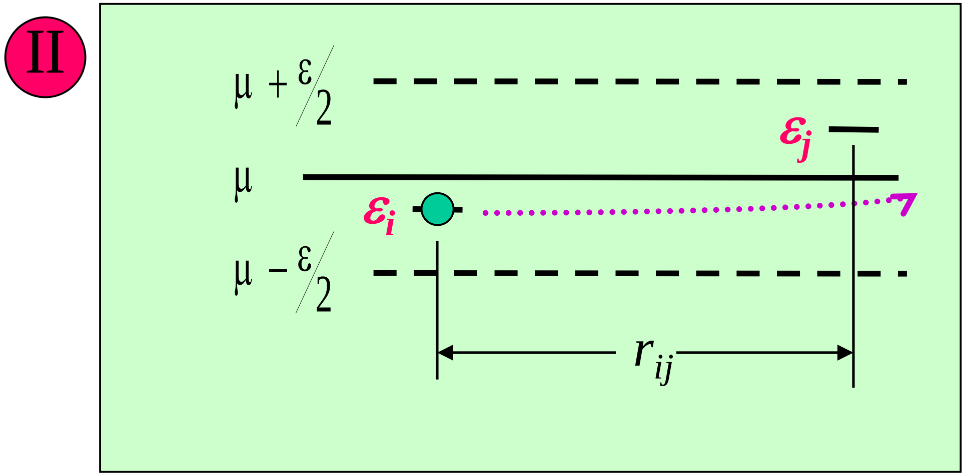
$$r_{ij} = [N(\varepsilon)]^{-1/2}, \qquad u_{ij} = \frac{2}{g_\mu^{1/2} a_B \varepsilon^{1/2}} + \frac{\varepsilon}{T},$$

$$\varepsilon_{\min} = \left(\frac{T}{g_\mu^{1/2} a_B} \right)^{2/3} = (T^2 T_{Mott})^{1/3}, \quad T_{Mott} = (g_\mu a_B^2)^{-1}$$

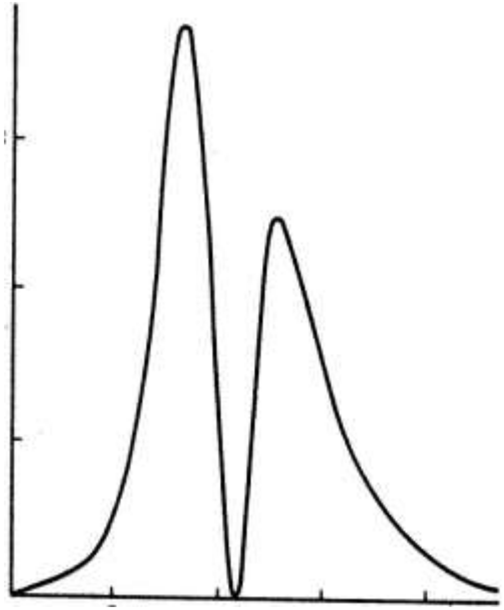
$$\boxed{\rho = \rho_0 \exp\left(T_{Mott}/T \right)^{1/3} \qquad (d = 2)}$$

Coulomb gap





$$\varepsilon_j - \varepsilon_i - {e^2}/{\kappa r_{ij}} \ge 0 \quad \rightarrow \quad {e^2}/{\kappa r_{ij}} \le \varepsilon$$



3D

$$N(\varepsilon) = r_{ij}^{-3} \le \left(\frac{\kappa\varepsilon}{e^2}\right)^3, \quad g(\varepsilon) = \frac{\partial N}{\partial \varepsilon} \propto \frac{\varepsilon^2 \kappa^3}{e^6}$$

2D

$$N(\varepsilon) = r_{ij}^{-2} \le \left(\frac{\kappa\varepsilon}{e^2}\right)^2, \quad g(\varepsilon) = \frac{\partial N}{\partial \varepsilon} \propto \frac{|\varepsilon|\kappa^2}{e^4}$$

Variable range hopping: Shklovskii-Efros

Coulomb gap

$$g(\varepsilon)=\left(\frac{\kappa}{e^2}\right)^d |\varepsilon|^{d-1}, \qquad g(0)=0$$

Number of states in ε - interval near Fermi level

$$N(\varepsilon)=\left(\frac{\kappa\varepsilon}{e^2}\right)^d$$

Same procedure as in the Mott case

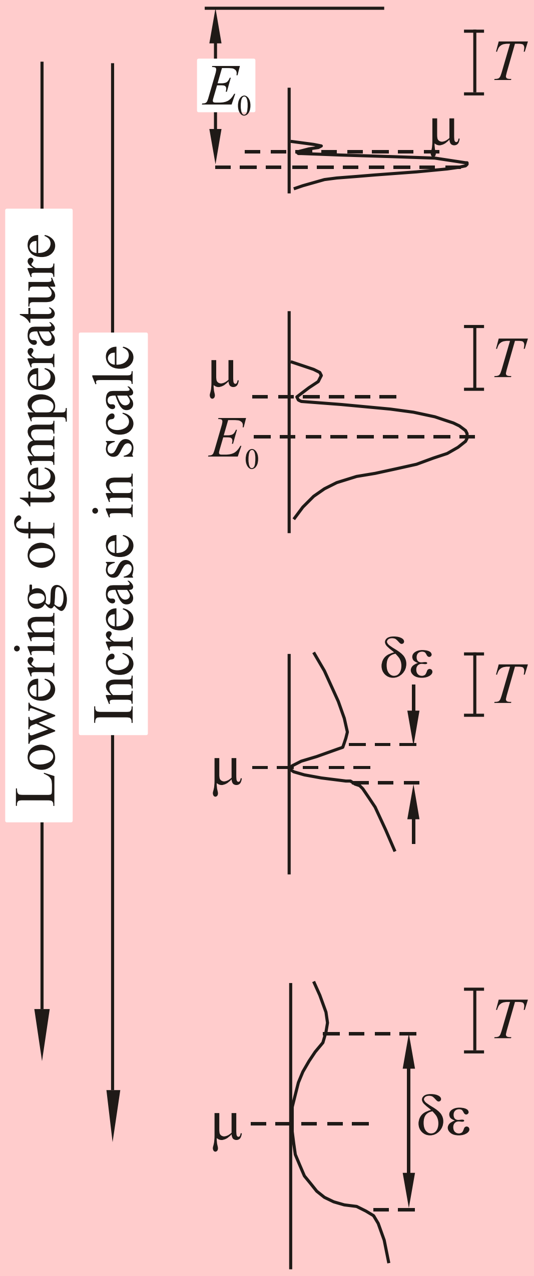
$$r_{ij}=[N(\varepsilon)]^{-1/d}=\frac{e^2}{\kappa\varepsilon}, \qquad u_{ij}=\frac{2}{a_B[N(\varepsilon)]^{1/d}}+\frac{\varepsilon}{T}=\frac{2e^2}{\kappa a_B\varepsilon}+\frac{\varepsilon}{T},$$

$$\varepsilon_{\min}=\left(\frac{2e^2T}{\kappa a_B}\right)^{1/2}=(TT_{ES})^{1/2}, \qquad T_{ES}=\frac{2e^2}{\kappa a_B}$$

resistance

$$\boxed{\rho=\rho_0\exp\left(T_{ES}/T\right)^{1/2} \qquad (d=3,2)}$$

Mechanisms of hopping conductivity



Thermal delocalization of carriers

$\exp\,(-E_0/T)$

Nearest-neighbor hopping

$\exp[-(\mu-E_0)/T]$

Variable-range hopping, Mott law ($T>\delta\varepsilon$)

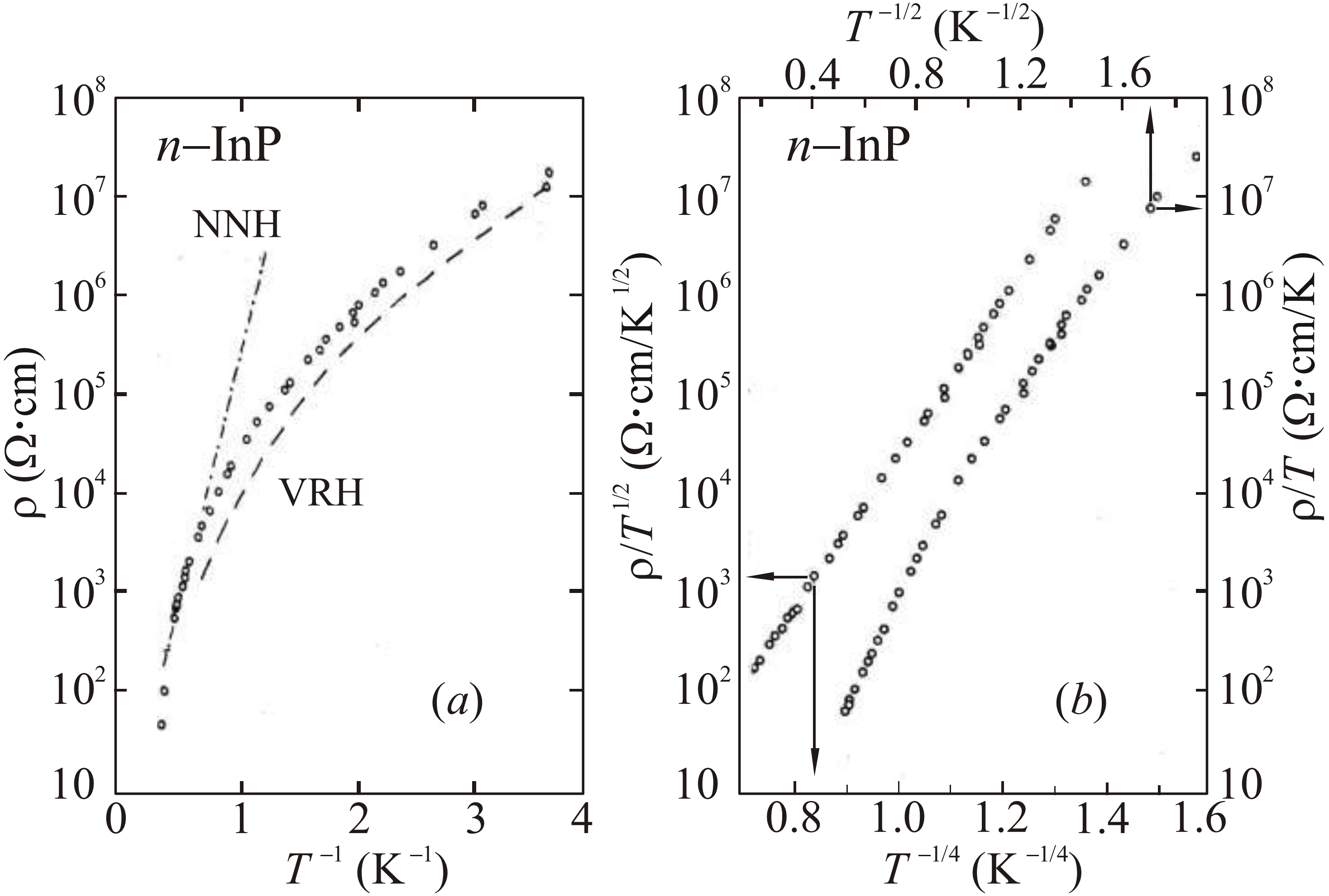
$\exp[-(T_M/T)^{1/4}]$

Variable-range hopping, Efros-Shklovskii law ($T<\delta\varepsilon$)

$\exp[-(T_{ES}/T)^{1/2}]$

Variable range hopping: experiment

temperature dependence, fitting by standard functions



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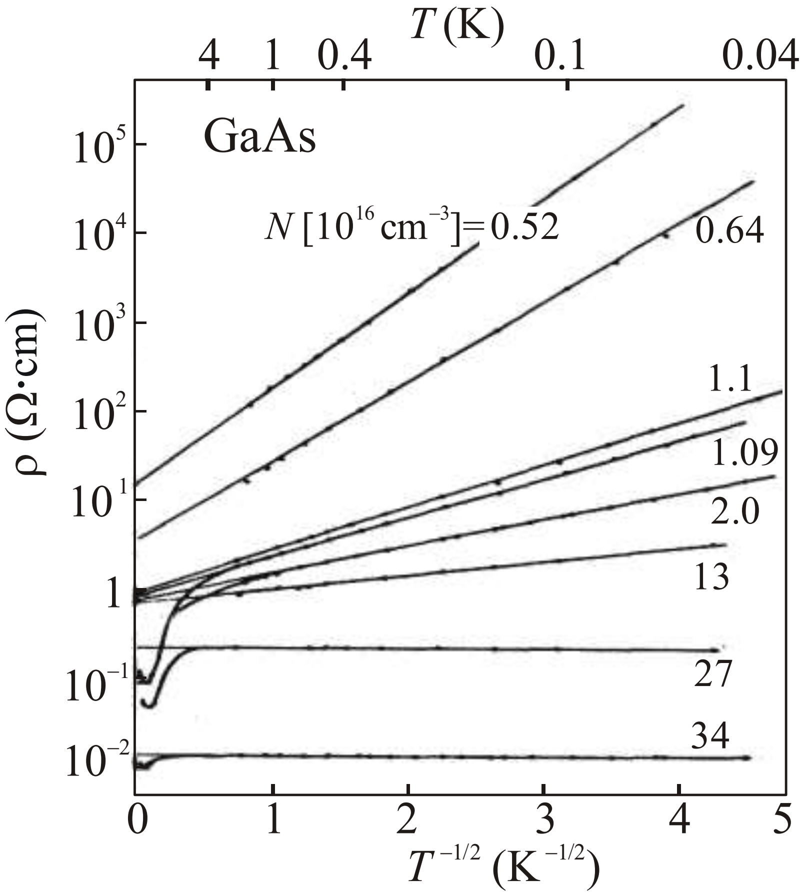
experiments



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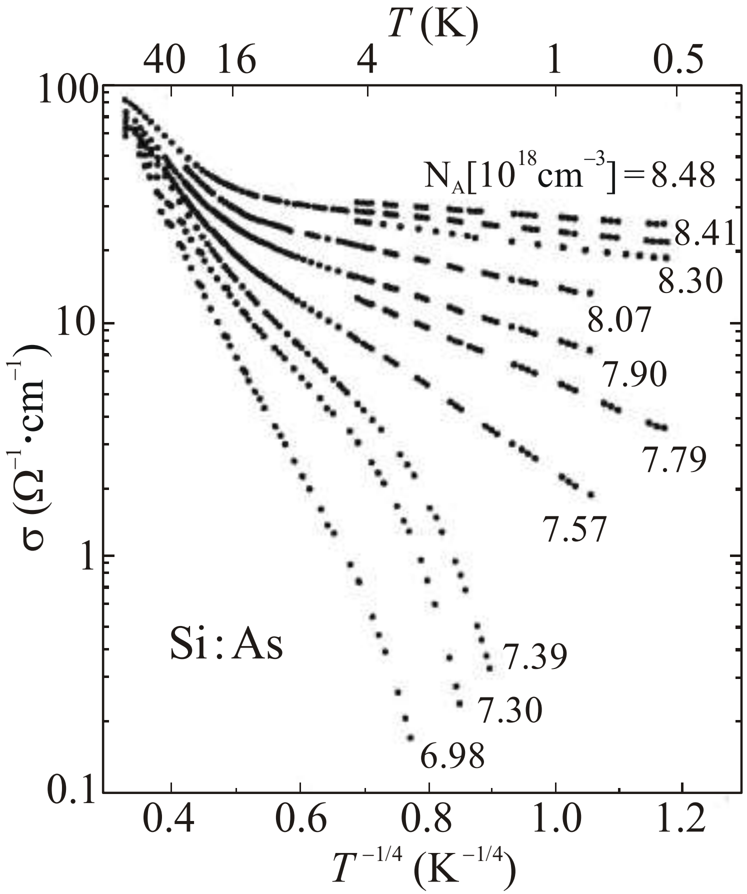

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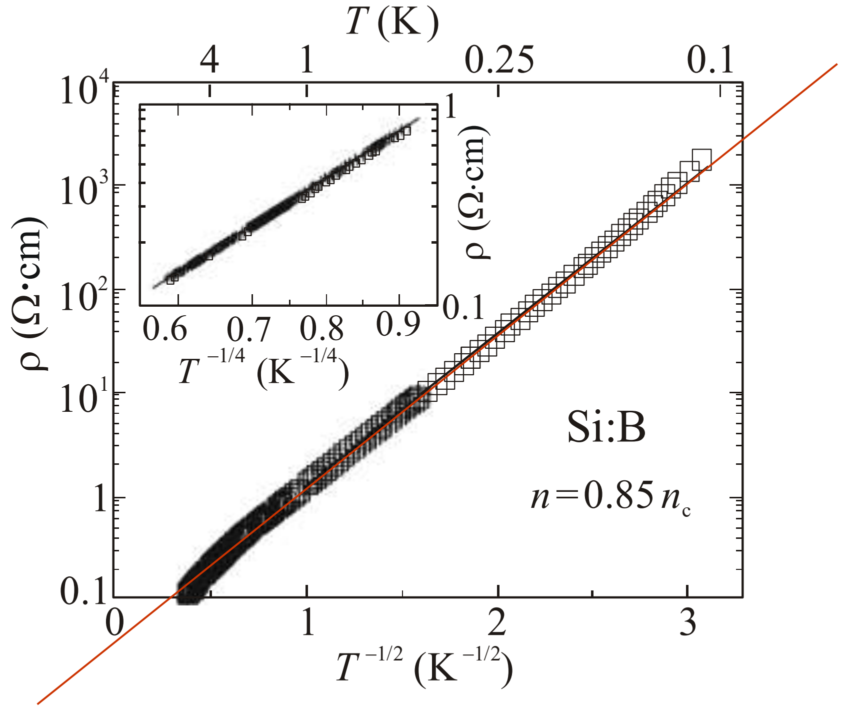
More experiments: Si:B



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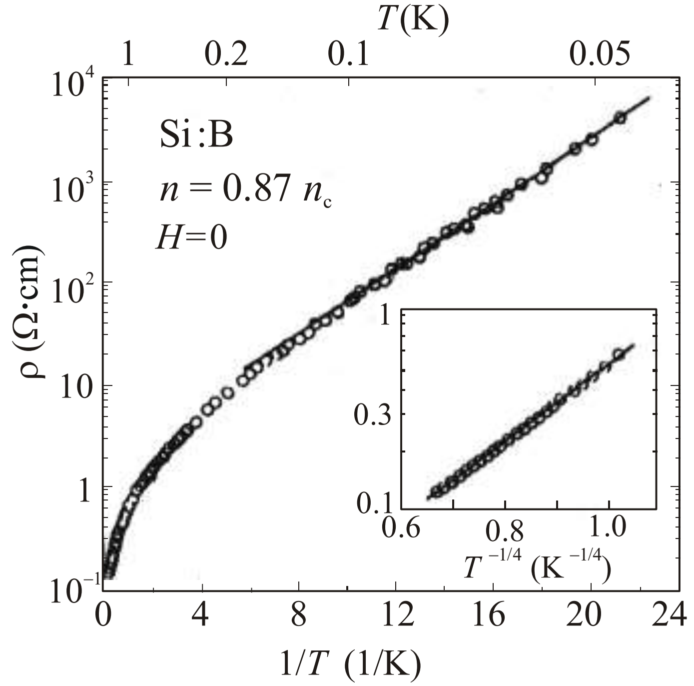
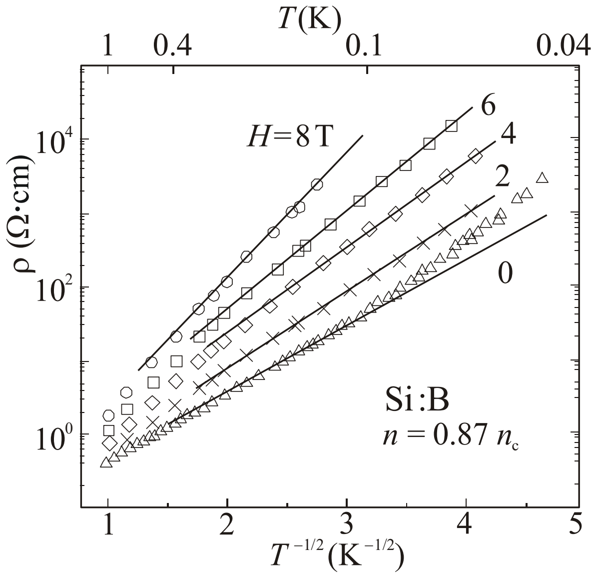
PRL **75,** 4266 (1995)

P. Dai, Y. Zhang, M.P. Sarachik, PRL **69,** 1804 (1992)





PHYSICAL REVIEW B VOLUME 52, NUMBER 8 15 AUGUST 1995-II

Electric-field activated variable-range hopping transport in $PrBa_2Cu_3O_{7-\delta}$

G. K. van Ancum, M. A. J. Verhoeven, D. H. A. Blank, and H. Rogalla
Department of Applied Physics, University of Twente, P.O. Box 217, NL-7500 AE Enschede, The Netherlands

(Received 25 January 1995)

We demonstrate the transport of charge carriers in $PrBa_2Cu_3O_{7-\delta}$ (PBCO) to be dependent both on the applied electric field and on the temperature. In our measurements we use inert noble-metal contacts on laser ablated and sputtered PBCO films. By applying the transmission line model we are able to separate the contact resistance from the PBCO resistance. The average hopping distance can be found by extending Mott's formula to field activation, and is found to be much greater than the dimensions of the PBCO unit cell. From the measurements in strong electric field a minimum hopping distance in the direction of the applied field of about 14 nm is determined, which we discuss in terms of localized states and intrinsic mixed valence of the Pr atoms in the PBCO film.

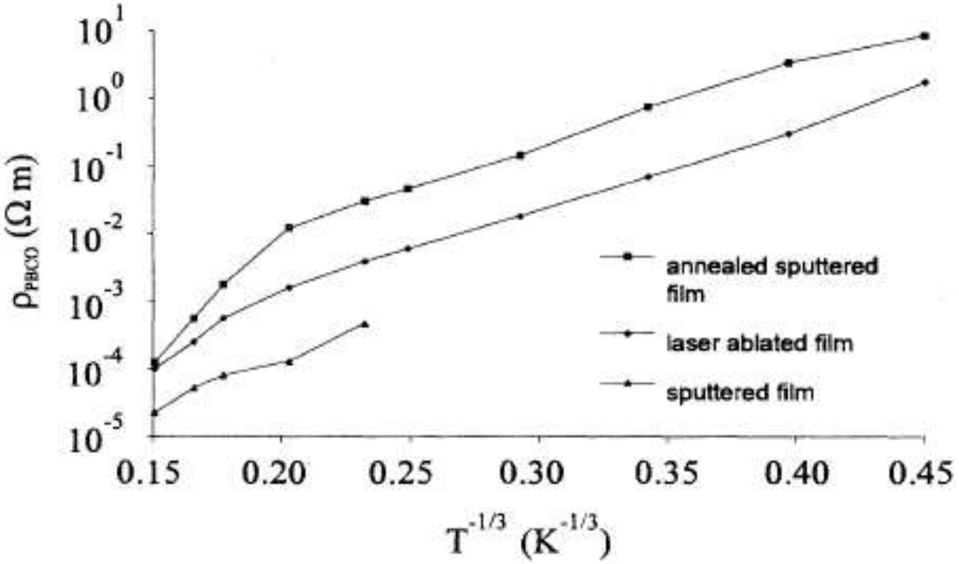


FIG. 3. The PBCO resistivity ρ_{PBCO} for laser ablated and sputtered films in zero electric field.

Magnetoresistance of $PrBa_2Cu_3O_{7-\delta}$ thin films

G. K. van Ancum, M. A. J. Verhoeven, D. H. A. Blank, and H. Rogalla
Department of Applied Physics, University of Twente, P.O. Box 217, NL-7500 AE Enschede, The Netherlands

(Received 26 June 1995)

Transport of charge carriers in $PrBa_2Cu_3O_{7-\delta}$ (PBCO) is often described by variable-range hopping (VRH). Until now the VRH mechanism was confirmed merely on the basis of a temperature dependence of the resistivity following Mott's law. In this article we show a positive magnetoresistance in PBCO thin films, depending exponentially on the applied magnetic field. This provides substantial additional evidence for a variable-range hopping transport mechanism. Both a strong-field and a weak-field magnetoresistance can be identified. At temperatures above 30 K we observe weak-field magnetoresistance, at 4.2 K we detect a transition from weak-field to strong-field magnetoresistance at a magnetic field of approximately 4.5 T. In the weak-field regime the radius of the localized wave function is only affected marginally by the applied magnetic field. In the strong-field regime the radius of the localized wave function decreases with increasing magnetic field. From the measurements in the strong-field regime we obtain an estimate for the two-dimensional density of localized states in the PBCO thin film of approximately 2×10^{13} 1/eVm^2.

$$a_H = a_0 / \ln\left(H \frac{k_B^2 \hbar^3}{m^2 e^3} \right) \quad \text{for } \lambda \leq a_0,$$

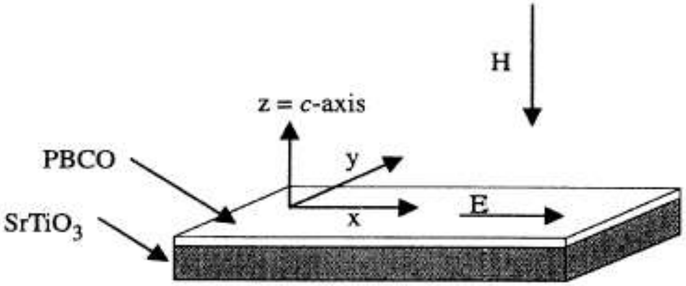


FIG. 1. Schematic drawing of the measurement setup.

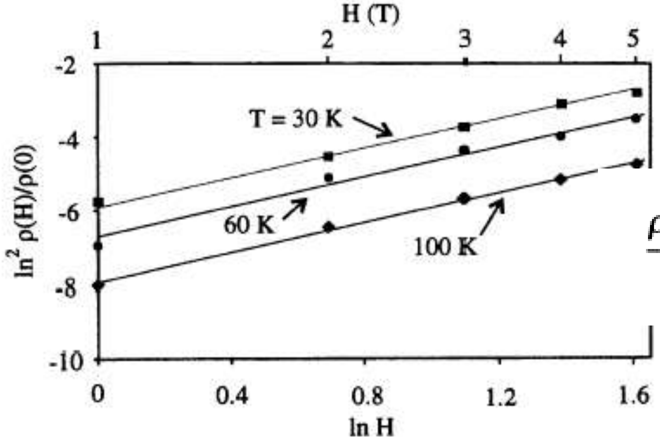


FIG. 2. Magnetoresistance at 100, 60, and 30 K (sample No. 1). The drawn lines represent weak-field dependence.

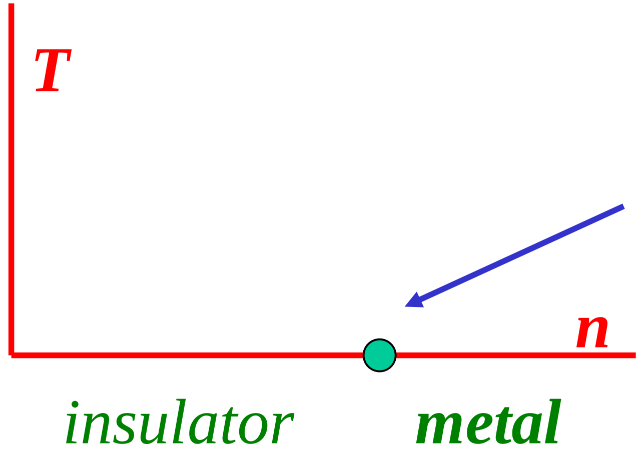
$$\frac{\rho_{\text{weak}}(H)}{\rho(0)} \propto \exp\left(\frac{H^2}{k_B T} \right).$$

Metal-Insulator transitions

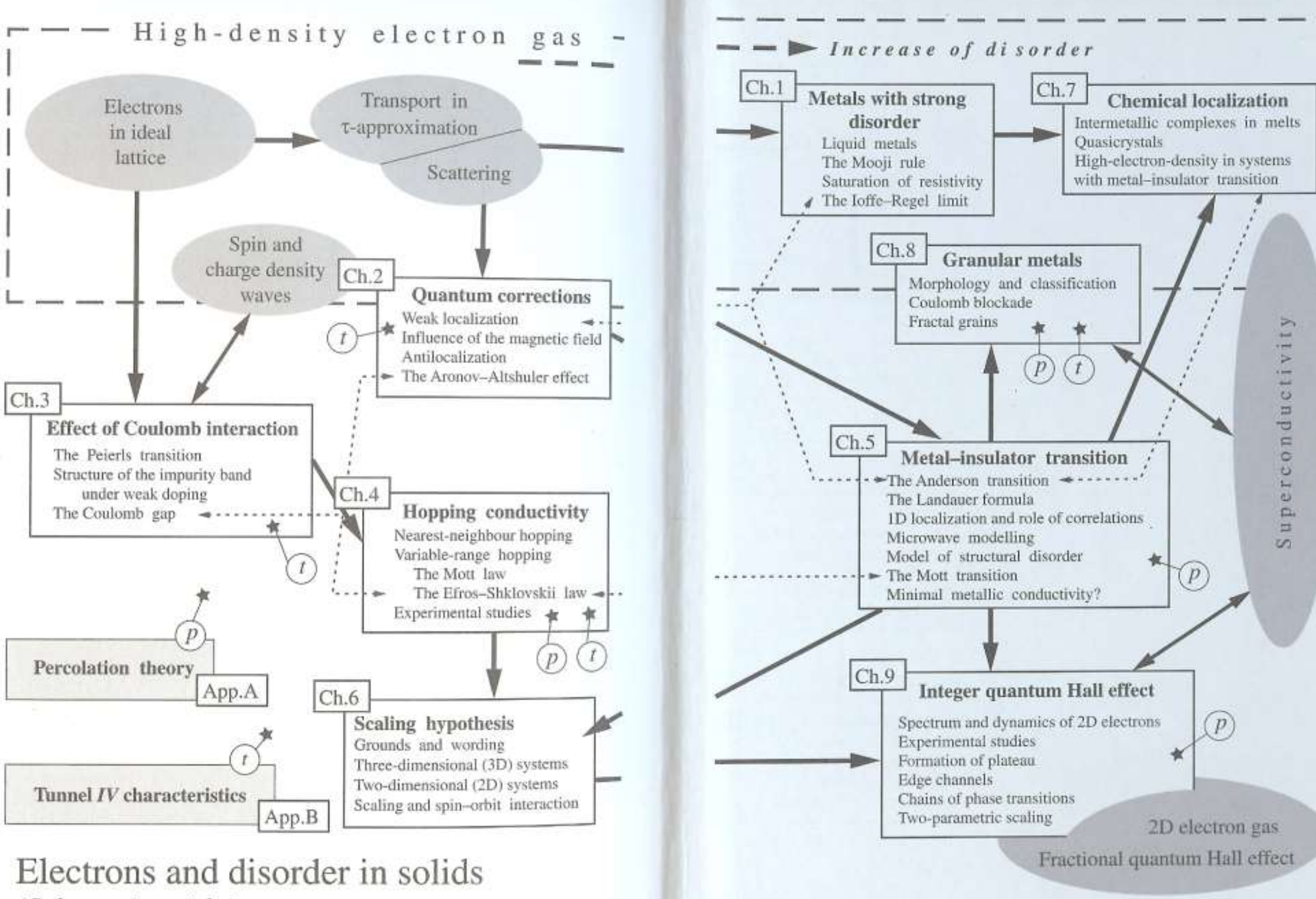
The definition: ***metal*** – $\sigma \neq 0$
insulator – $\sigma = 0$ $\longrightarrow$ ***at*** $T = 0$

'Driving force'

- **disorder: Anderson transition**
- **interactions: Mott transition**



Isolated point at the phase diagram



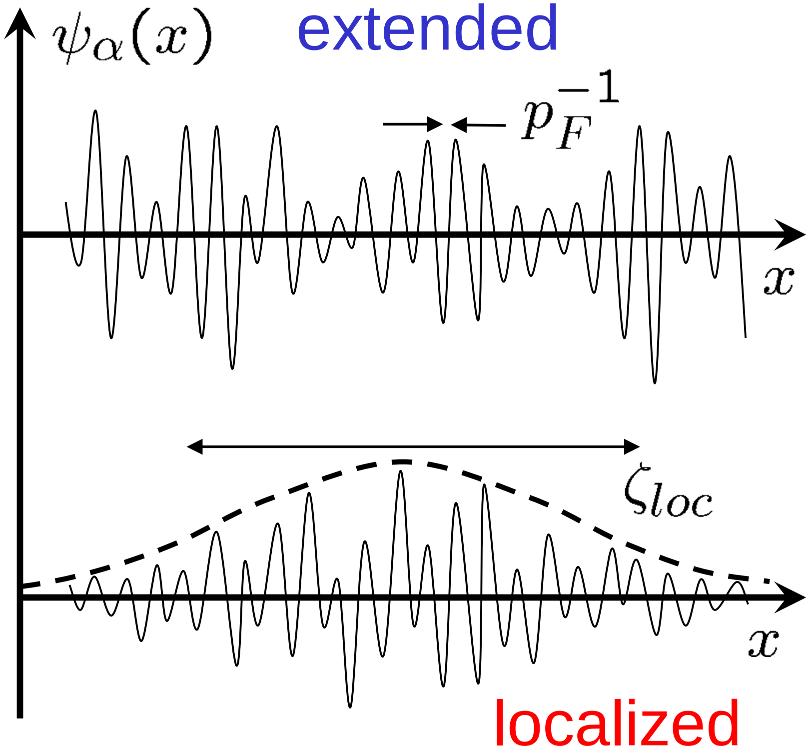
Electrons and disorder in solids

(Schematic guide)

Localization of single-particle wave-functions:

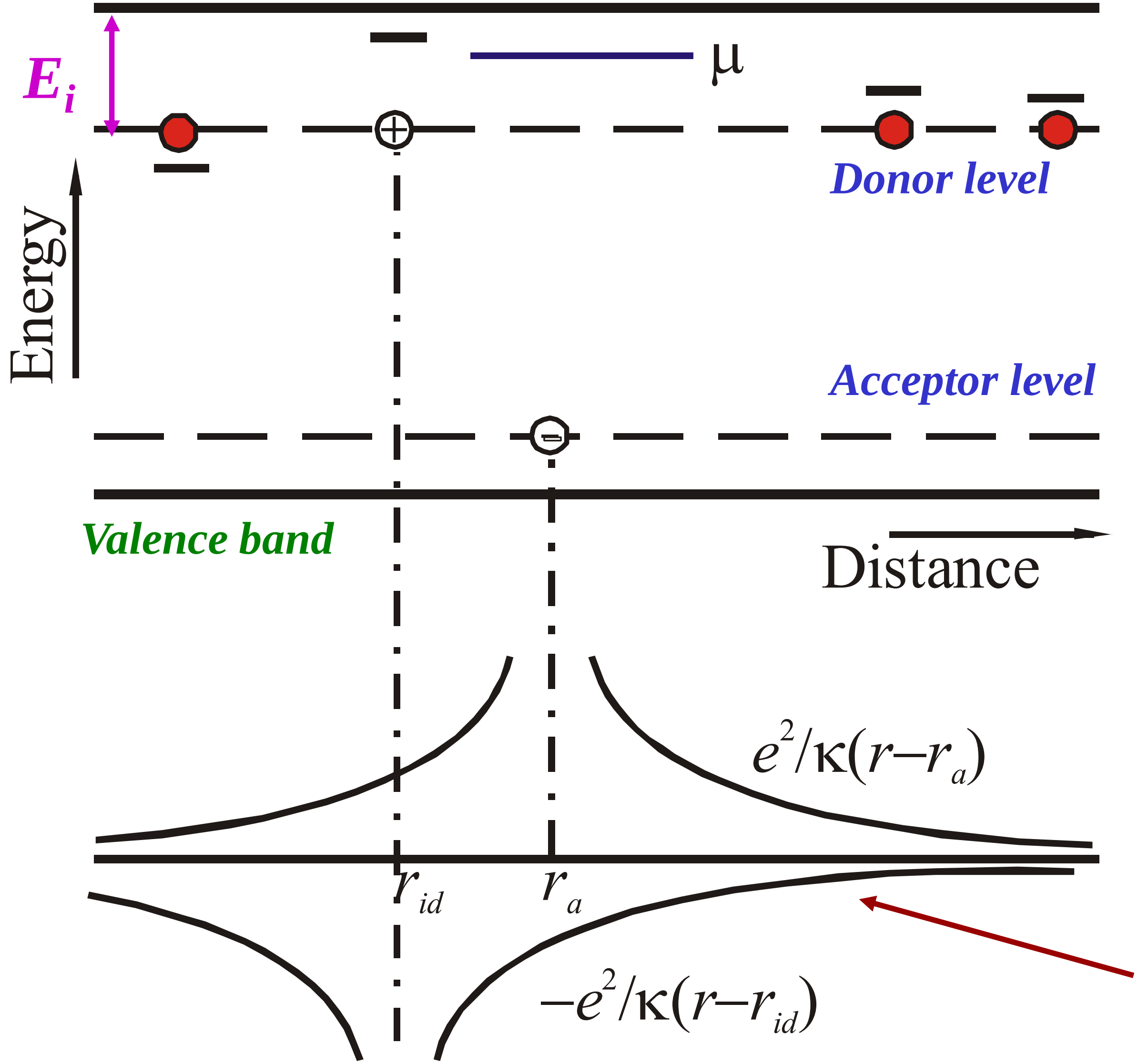
$$\left[-\frac{\boldsymbol{\nabla}^2}{2m}+U(\boldsymbol{r})-\epsilon_F\right]\psi_\alpha(\boldsymbol{r})=\xi_\alpha\psi_\alpha(\boldsymbol{r})$$

Disorder



Impurity band: we assumed $Na_B^3 \ll 1 \Rightarrow$

direct overlap of wave functions can be neglected, conduction band is not formed => insulator at T=0 (only thermally activated hopping is possible)



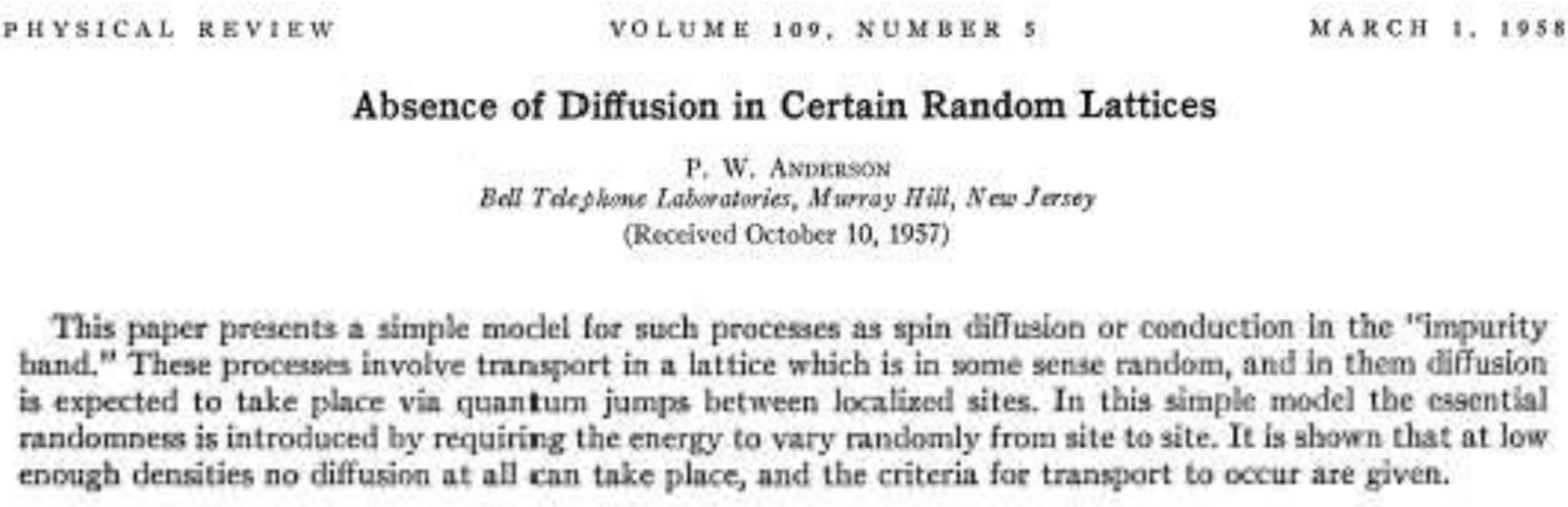
Question:

What happens if we increase the parameter

$$Na_B^3$$

Impurity band is formed due to electric fields of charged impurities

Quantum particle in random potential

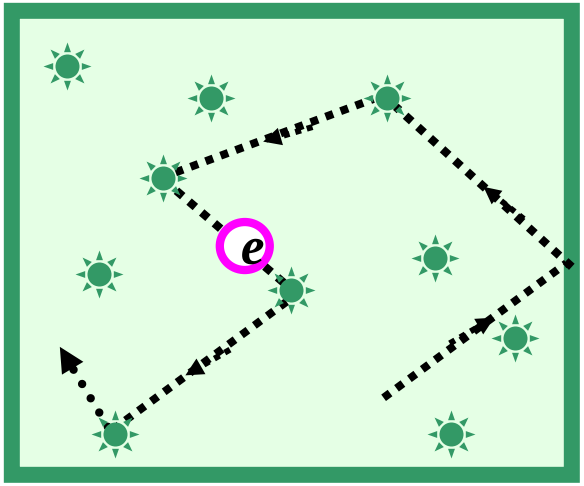


PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

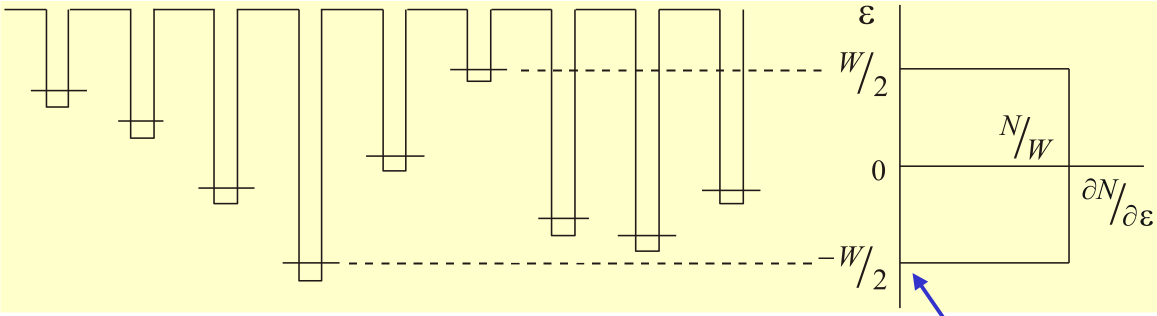
P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.



Anderson transition

$$\frac{\partial N}{\partial \varepsilon} = \begin{cases} N/W & \text{for } |\varepsilon| < W/2 \\ 0 & \text{for } |\varepsilon| > W/2 \end{cases}$$

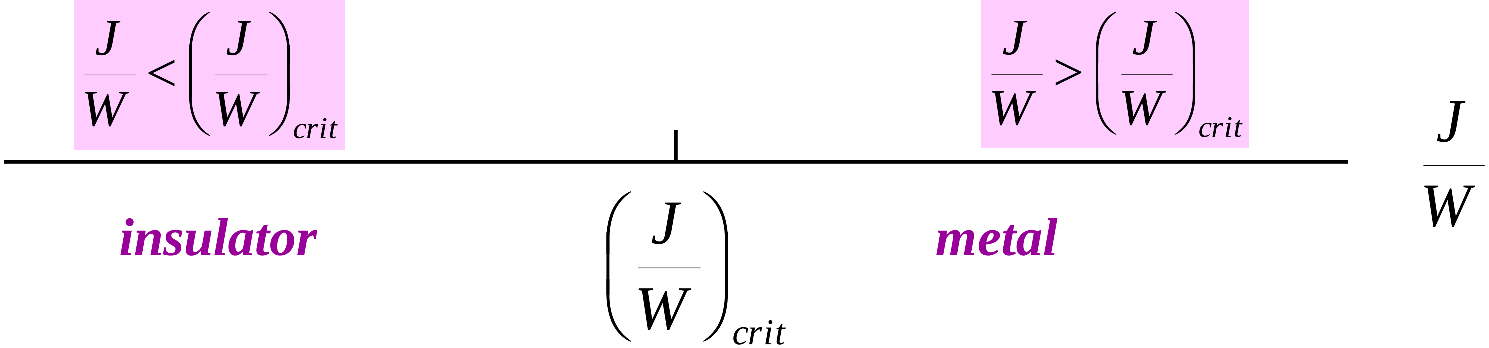


Transfer integral

$$J = \int \psi_1^* \hat{H} \psi_2 d^3 r \propto \exp\left(-\frac{r_{12}}{a_B}\right) = E_0 \exp\left(-\frac{1}{a_B n^{1/3}}\right)$$

width W characterises disorder

ratio J/W - the main parameter of the problem



$$\frac{J}{W} < \left(\frac{J}{W}\right)_{crit}$$

$$\frac{J}{W} > \left(\frac{J}{W}\right)_{crit}$$

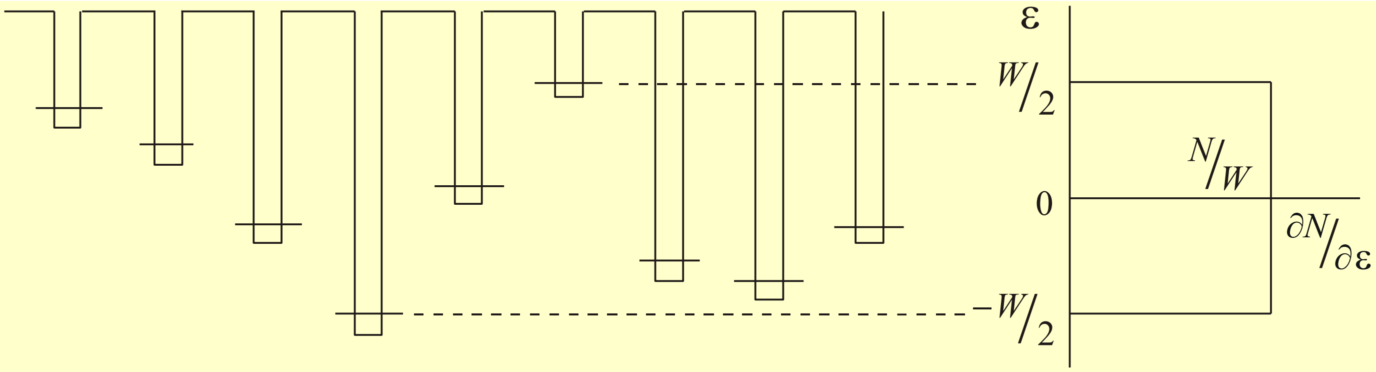
$$\frac{J}{W}$$

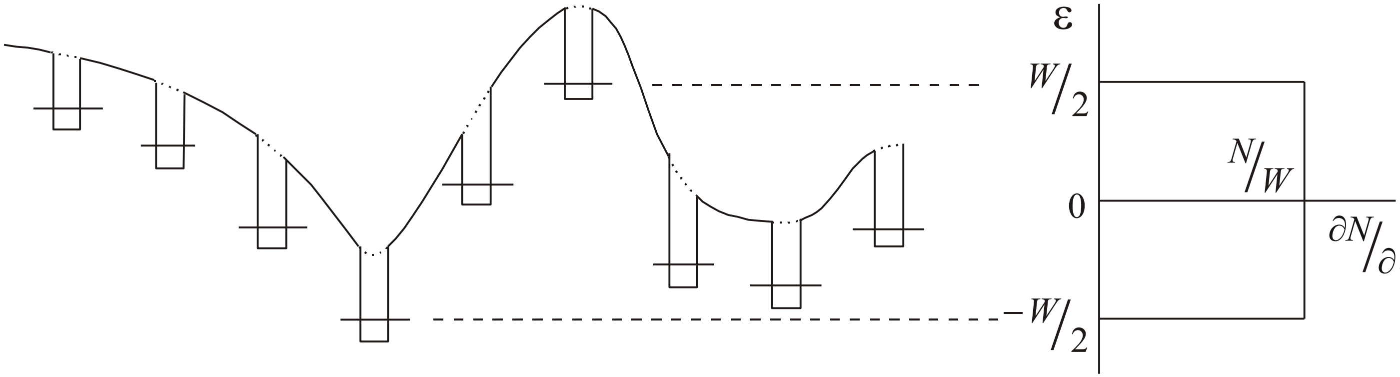
insulator

$$\left(\frac{J}{W}\right)_{crit}$$

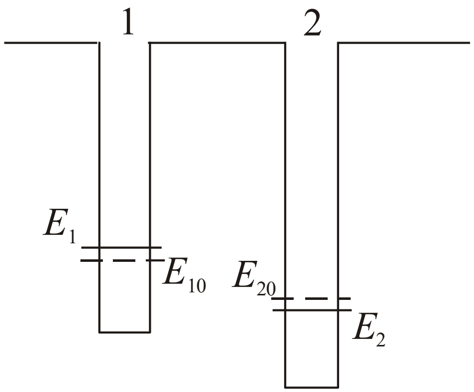
metal

Anderson transition: different representations



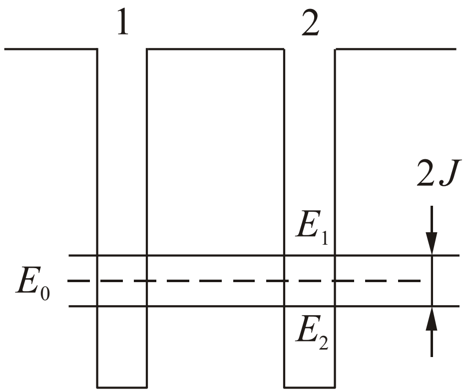


Anderson transition



If $E_{01} \neq E_{02}$
then level shift is

$$\Delta E \propto e^{-2r/a_B} \propto J^2$$



If $E_{01} = E_{02} = E_0$,
then level shift is

$$\Delta E \sim J \propto e^{-r/a_B}$$

(resonant nodes)

The parameter J/W is the fraction of resonant nodes

$$J = E_0 \exp\left(-\frac{1}{a_B n^{1/3}}\right) \qquad a_B n^{1/3}{}_{crit} = -\left(\ln \frac{c_a W}{E_0}\right)^{-1}, \quad c_a = \left(\frac{J}{W}\right)_{crit}$$

Delocalized states first appear at the energy band center

usual 'band insulator': the density of states at the Fermi level is zero
Anderson insulator: the density of states at the Fermi level is nonzero

Mott transition

3 length scales:

$$n^{-1/3}$$

average e-e distance

$$a_B = \frac{\kappa\hbar^2}{m^* e^2}$$

Bohr radius

$$r_e = \left(\frac{4m^* e^2 n^{1/3}}{\kappa\hbar^2} \right)^{-1/2}$$

screening length

are related

$$r_e = \frac{1}{2}\left(a_B n^{-1/3} \right)^{1/2}$$

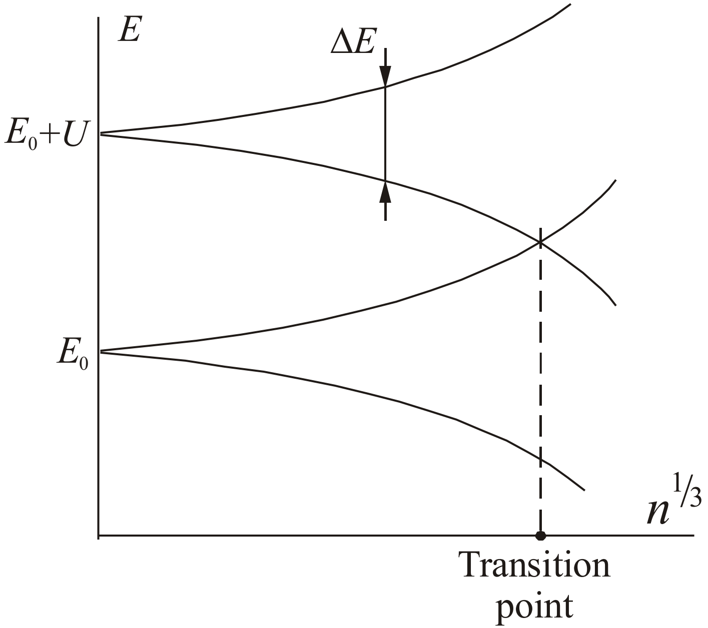
$r_e > a_B$ – *insulator*

$r_e < a_B$ – *metal*

Mott transition occurs if

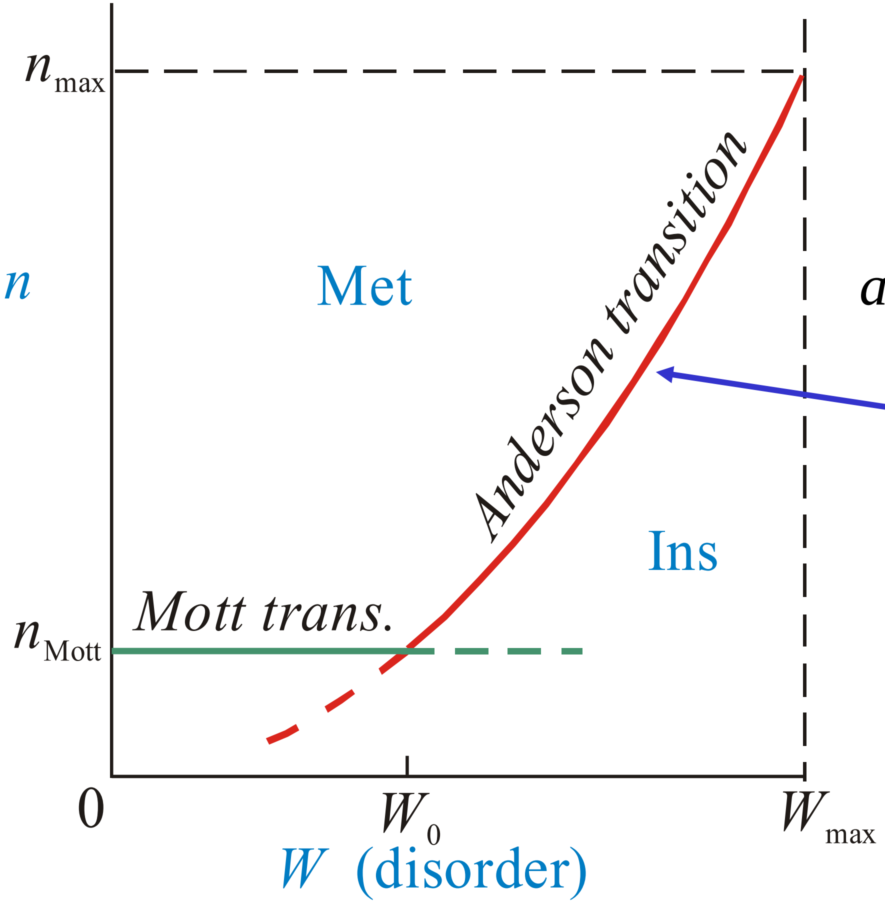
$$r_e = a_B$$

$$a_B n_c^{1/3} = 0.25$$



Hubbard model

Phase diagram «disorder – concentration» at $T = 0$



$$a_B n^{1/3} = -\left(\ln \frac{c_a W}{E_0}\right)^{-1}, \quad c_a = \left(\frac{J}{W}\right)_{crit}$$

assuming $\boldsymbol{a_B}$ = const

2 sources of localization: disorder and e-e interaction

Mott vs Anderson

MOTT TRANSITION

Width ΔE is proportional to

$J = E_0 \exp(-1/a_B n^{1/3})$

$\boldsymbol{\Delta E = U}$

$a_B n^{1/3} = -(\ln c_m U/E_0)^{-1}$

$\boxed{\boldsymbol{a_B n_c^{1/3} = 0.25}}$ *Mott relation*

$a_B n^{1/3} = -(\ln c_a W/E_0)^{-1}$

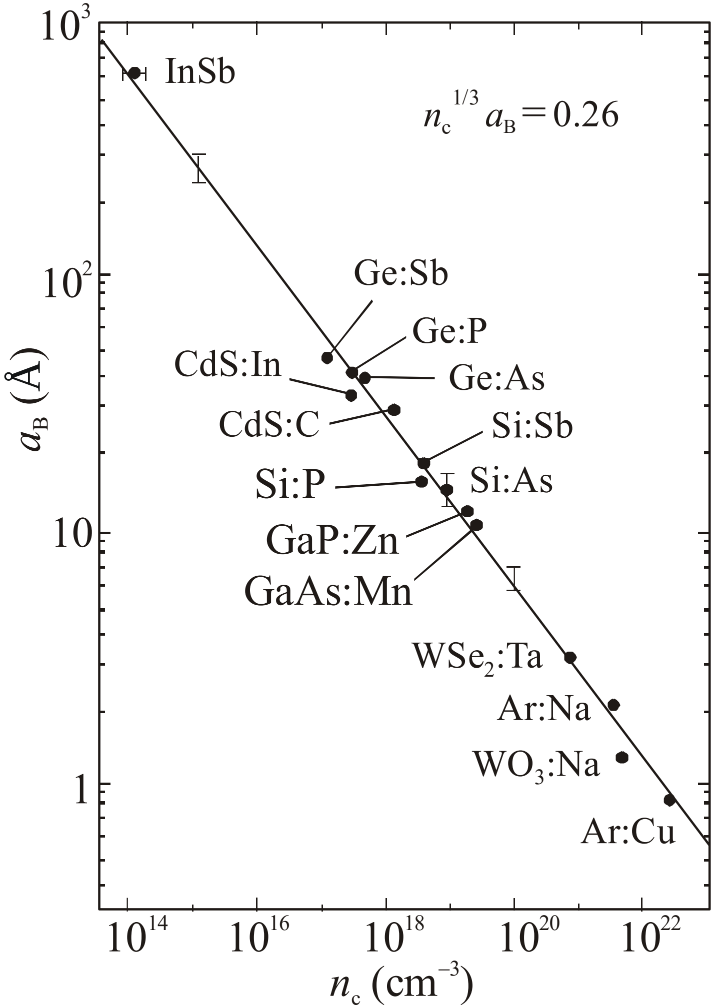
Transfer integral

$J = E_0 \exp(-1/a_B n^{1/3})$

enters parameter W/J

$\boldsymbol{W/J = \mathrm{const}}$

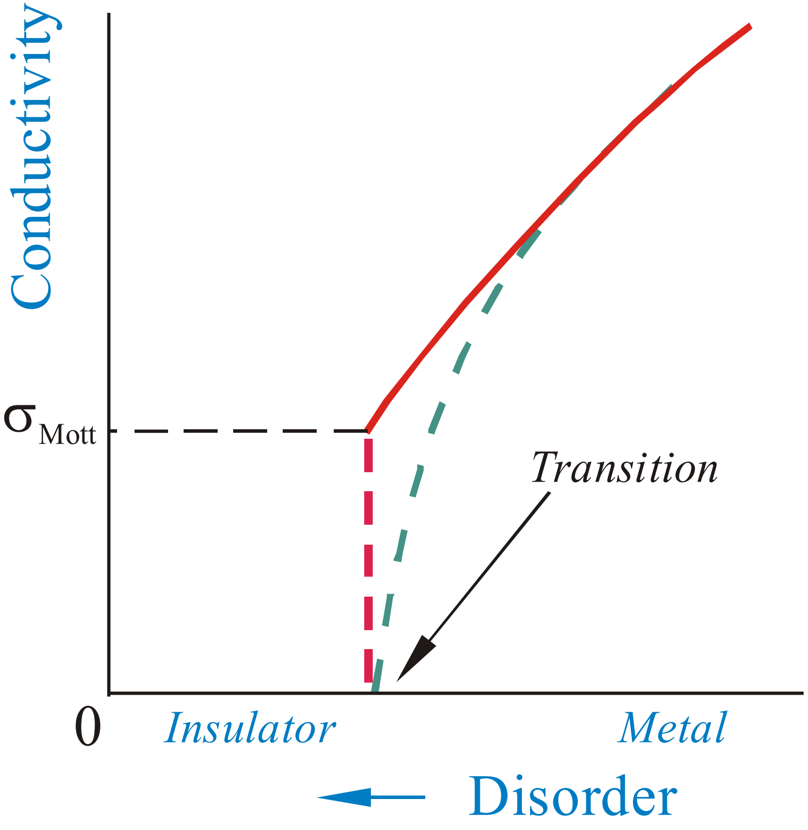
ANDERSON TRANSITION



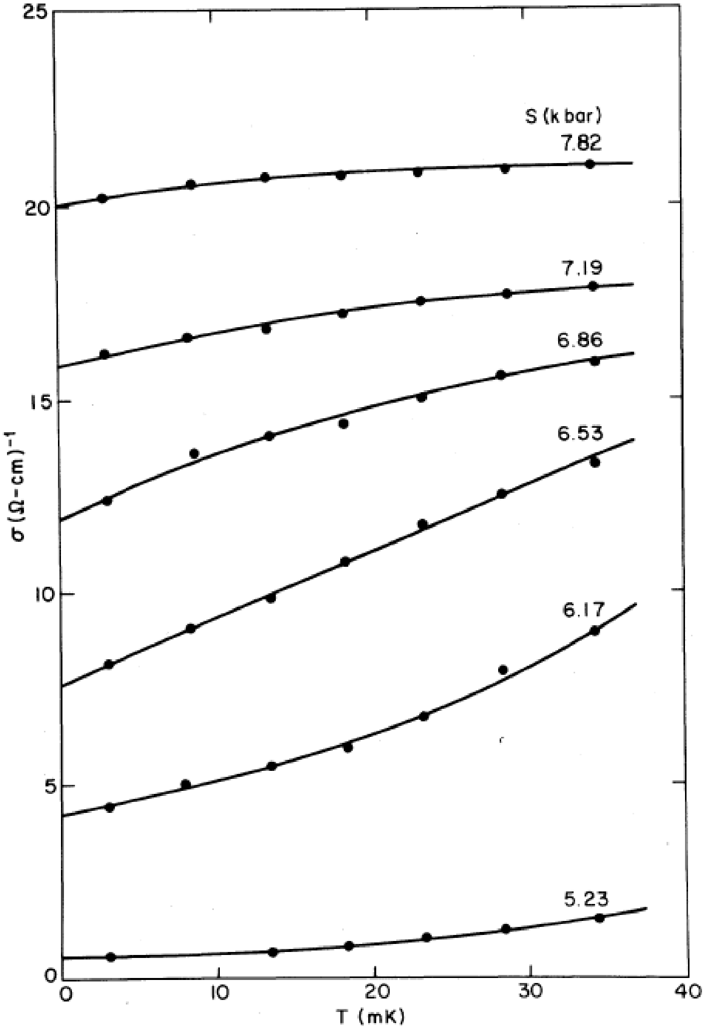
P.P. Edwards, M. J. Sienko,
*Phys.Rev. B **17**, 2575 (1978)*

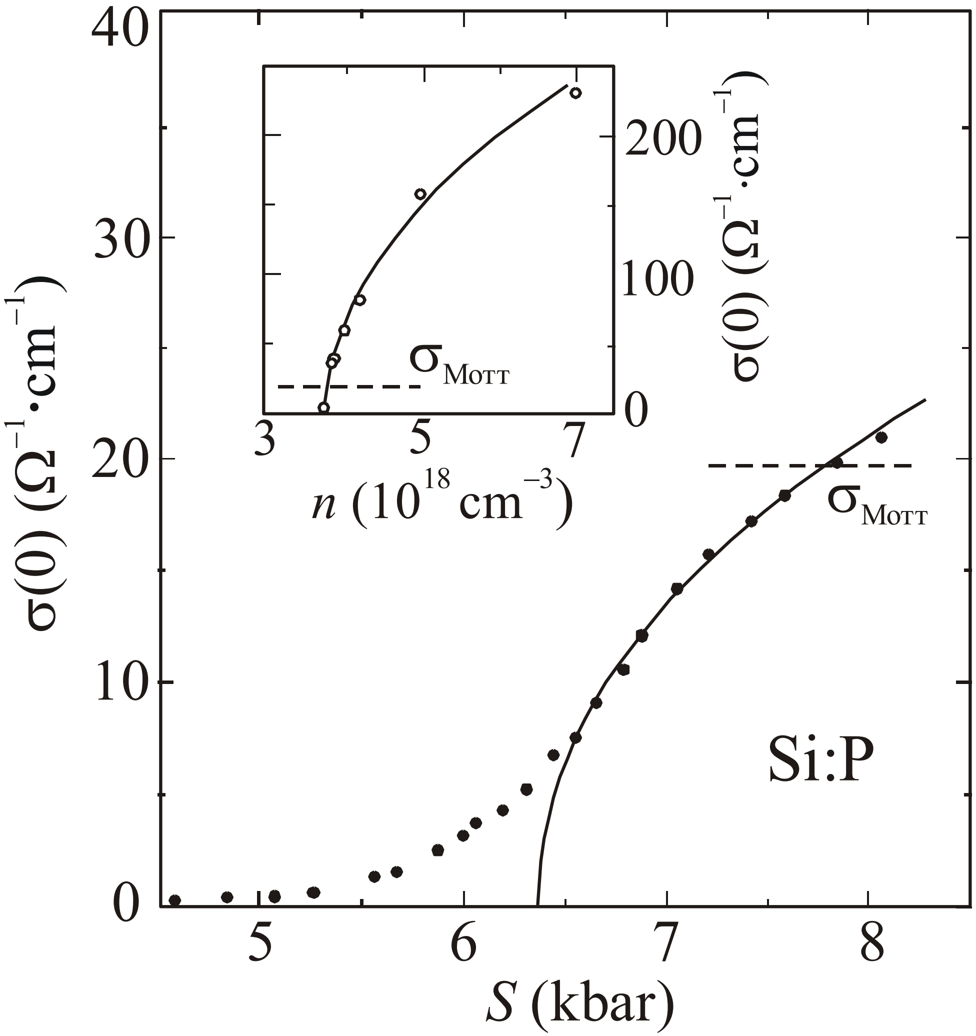
Minimum metallic conductivity ?

$$\sigma = \frac{ne^2 l}{\hbar k_F} \longrightarrow \sigma = A\frac{e^2}{\hbar}n^{1/3}(k_F l) \geq A\frac{e^2}{\hbar}n^{1/3}$$



Metal-insulator: a second order phase transition





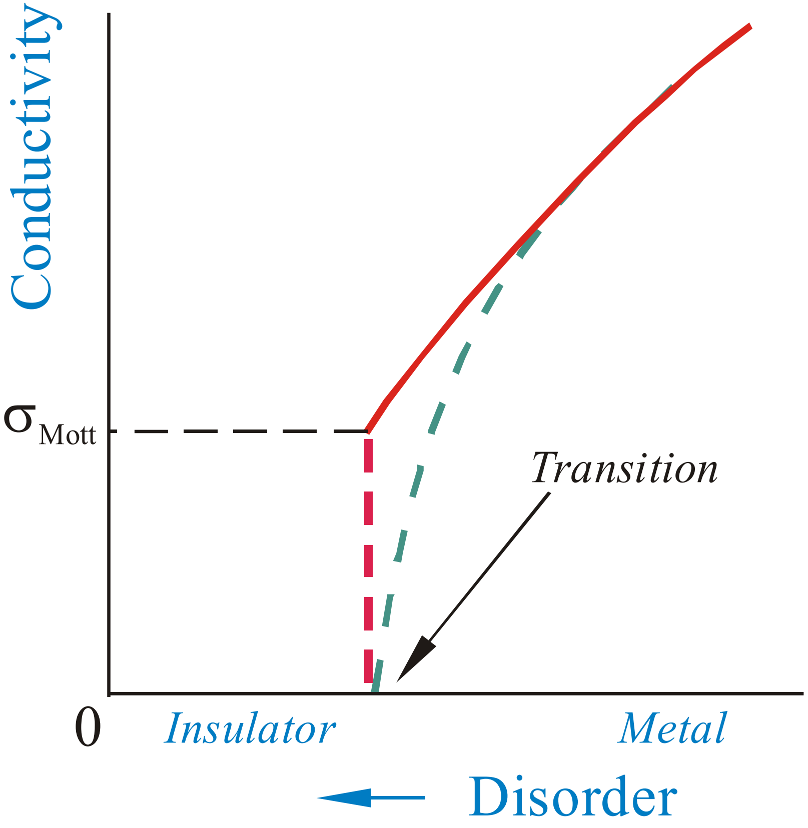
M. A. Paalanen, T.F. Rosenbaum, G.A. Thomas, R.N.Bhatt
*Phys.Rev.Lett.**48**, 1284 (1982)*

Scaling theory of localization

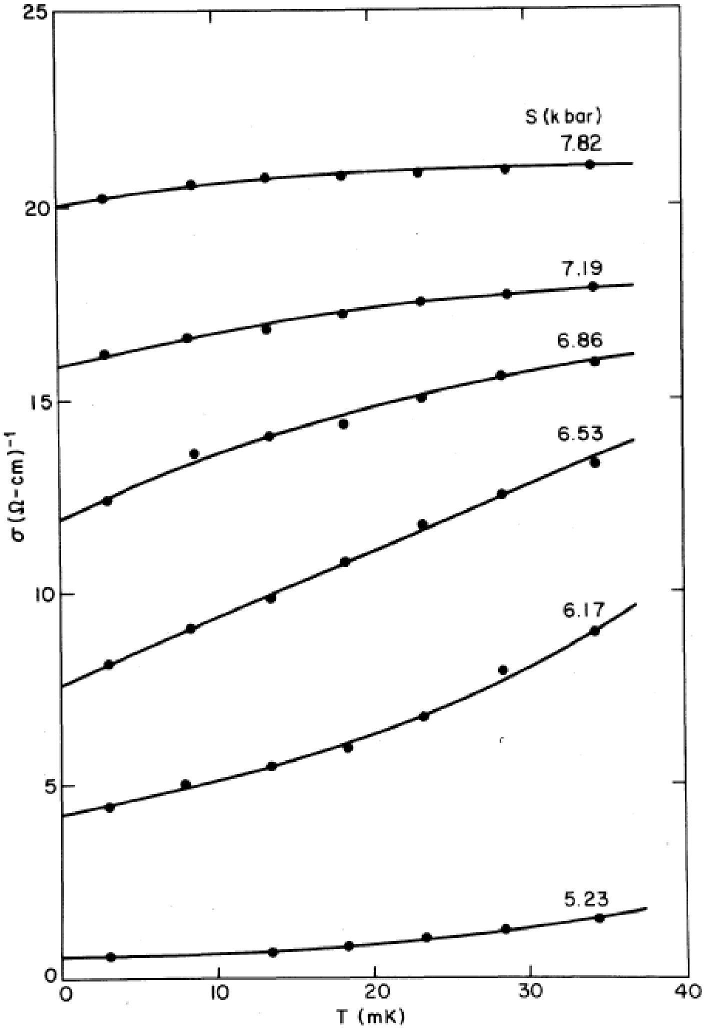
E.Abrahams, P.W.Anderson, D.C.Licciardello, and T.W.Ramakrishnan,
Phys.Rev.Lett. 42, 673 (1979)

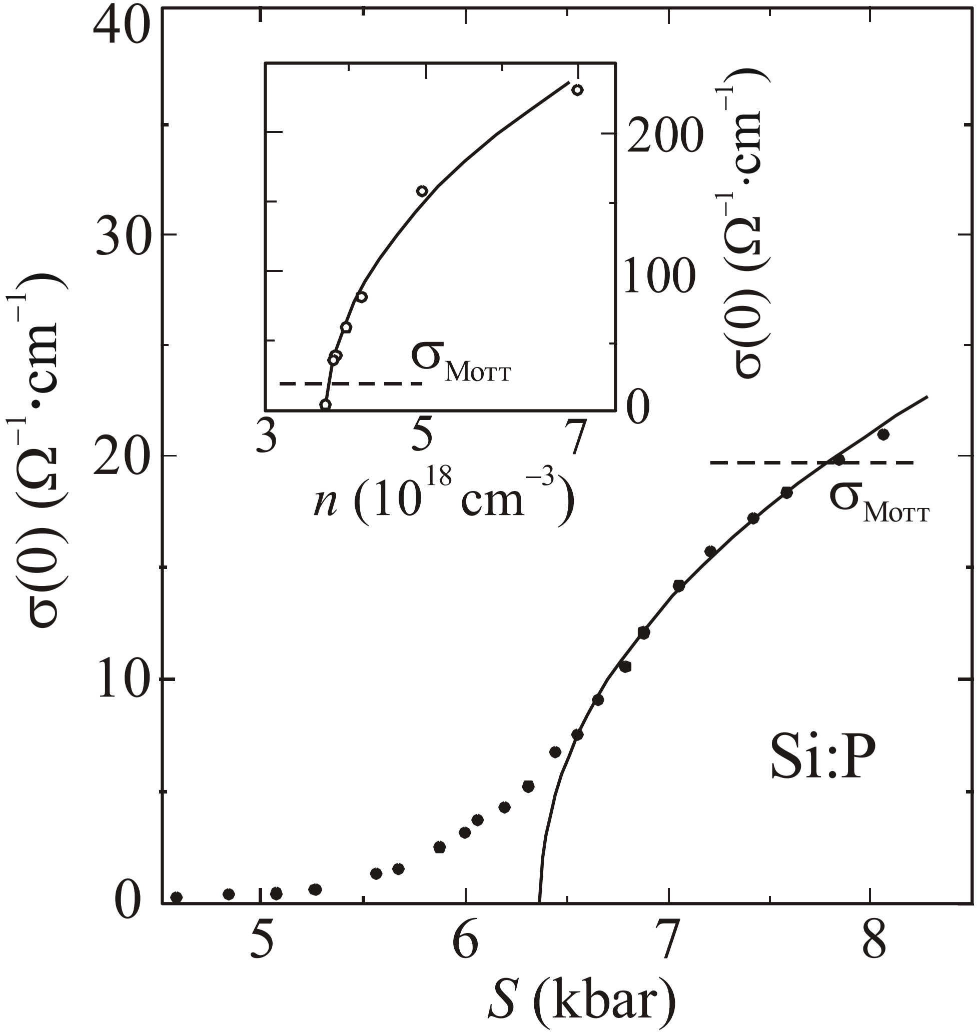
Minimum metallic conductivity ?

$$\sigma = \frac{ne^2 l}{\hbar k_F} \longrightarrow \sigma = A\frac{e^2}{\hbar}n^{1/3}(k_F l) \geq A\frac{e^2}{\hbar}n^{1/3}$$



Metal-insulator: a second order phase transition



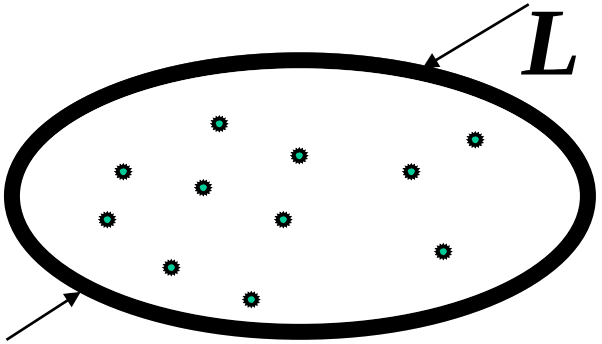


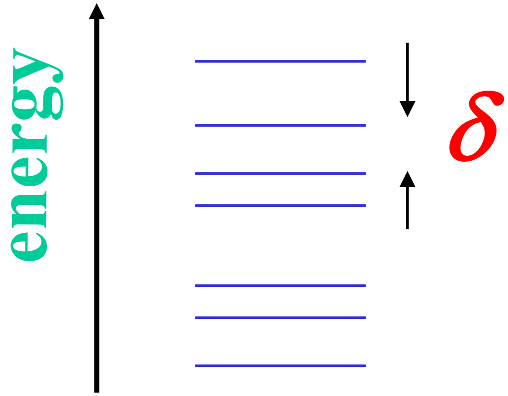
M. A. Paalanen, T.F. Rosenbaum, G.A. Thomas, R.N.Bhatt
*Phys.Rev.Lett.**48**, 1284 (1982)*

Energy scales (*Thouless, 1972*)

1. Mean level spacing

$$\delta = 1/gL^d$$





L is the system size;

d is the number of dimensions

2. Thouless energy

$$E_T = \hbar D / L^2$$

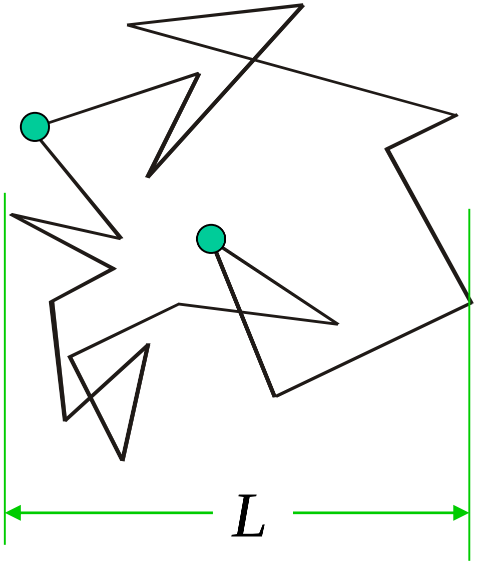
D is the diffusion const

E_T *has a meaning of the* ***inverse diffusion time*** *of the traveling through the system*

$$y = E_T / \delta$$

dimensionless
Thouless
conductance

Dephasing due to *ee* – interaction



ballistic $r \sim v_F t$

diffusive $r \sim l\sqrt{t/\tau} \sim v_F\sqrt{t\tau}$

phase $\exp(i\varphi) = \exp[i(\varepsilon_i/\hbar)t], \quad \Delta\varphi = (\Delta\varepsilon/\hbar)t$

dephasing time

$$\left.\begin{matrix}\tau_{ee} \sim \hbar/\Delta\varepsilon \\ \Delta\varepsilon \sim T\end{matrix}\right\} \quad \boxed{\tau_{ee} \sim \hbar/T}$$

dephasing length

$$\boxed{L_{ee} \approx \sqrt{D\tau_{ee}} \approx \sqrt{\frac{\hbar D}{T}}}$$

Electrons diffuse and keep their coherence during time τ_{ee}

typical e-e distance during this time is L_{ee} .

Thouless energy

(a) We use the expression for the dephasing length

$$\boxed{L_{ee} \approx \sqrt{D\tau_{ee}} \approx \sqrt{\frac{\hbar D}{T}}}$$

(b) Replace *Lee* by the system size *L* and replace temperature *T* by energy *E*

(c) We define the corresponding energy scale, ***the Thouless energy***

$$\boxed{E_T = \hbar D / L^2}$$

E_T has a meaning of the inverse diffusion time of the traveling through the system.
It determines the "phase coherent" energy interval around Fermi energy for a given system size L

Einstein relation for electric conductivity σ

Conductivity

$$\sigma = e^2 g D$$

Conductance

$$Y = \sigma L^{d-2}$$

for a cubic sample of the size L, measured in 1/Ohm

$$Y = \frac{e^2}{\hbar} (g L^d) \frac{\hbar D}{L^2} = \frac{e^2}{\hbar} y$$

$$y(L) = \frac{\hbar D / L^2}{1 / g L^d} = \frac{\text{Thouless energy}}{\text{mean level spacing}}$$

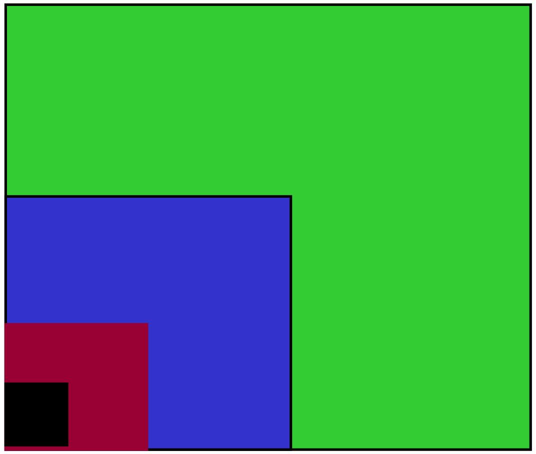
Dimensionless Thouless conductance

Scaling theory of Localization

Abrahams, Anderson, Licciardello and Ramakrishnan (1979)

$$y = E_T / \delta$$

Dimensionless Thouless conductance



$$L = 2L = 4L = 8L \ldots.$$

without quantum corrections:

$$E_T \propto L^{-2} \qquad \delta \propto L^{-d} \qquad \Rightarrow y \propto L^{d-2}$$

$$E_T \qquad E_T \qquad E_T \qquad E_T$$

$$\delta \rightarrow \delta \rightarrow \delta \rightarrow \delta$$

$$y \rightarrow y \rightarrow y \rightarrow y$$

In general: universal scaling

$$\boxed{\frac{d(\ln y)}{d(\ln L)} = \beta(y)}$$

Scaling hypothesis

control parameter: conductance *y*

conductivity	**σ [Ω^{-1}cm^{2-d}]**
Conductance	**Y [Ω^{-1}]**
Dimentionless conductance	**y**

$$Y = \sigma L^{d-2} \qquad y = \frac{Y}{e^2/\hbar} = \frac{\sigma L^{d-2}}{e^2/\hbar}$$

L – ***size of cube***

Derivation of scaling equation

L is a sample size

Let's perform the scaling $L \to qL$, $q = 1 + \alpha$, $\alpha \ll 1$

Scaling hypothesis:

The conductance y(L) is the control parameter

which determines the variation of y(L) as a function of L

$$y(qL) = f\,[q, y(L)]$$

$$y(L+\alpha L) = f\,[1+\alpha, y(L)]$$

Zero order in α: $y(L) = f\,[1, y(L)]$

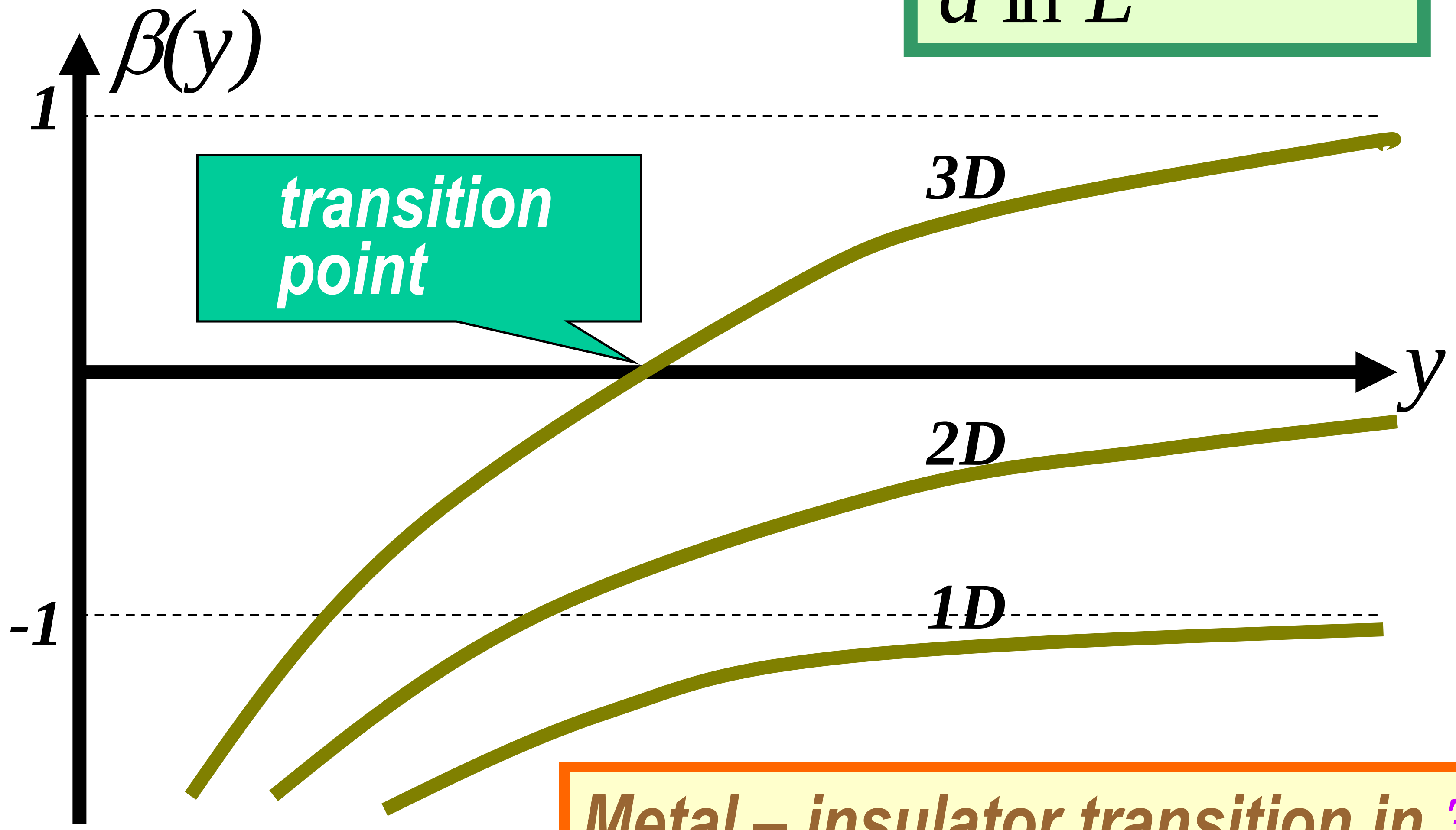
First order in α : $\alpha L(dy/dL) = \alpha\,(df/dq)$ at $q = 1$

Where $\beta(y) = (df/dq)/y$ at $q=1$

$$\boxed{\frac{d(\ln y)}{d(\ln L)} = \beta(y)}$$

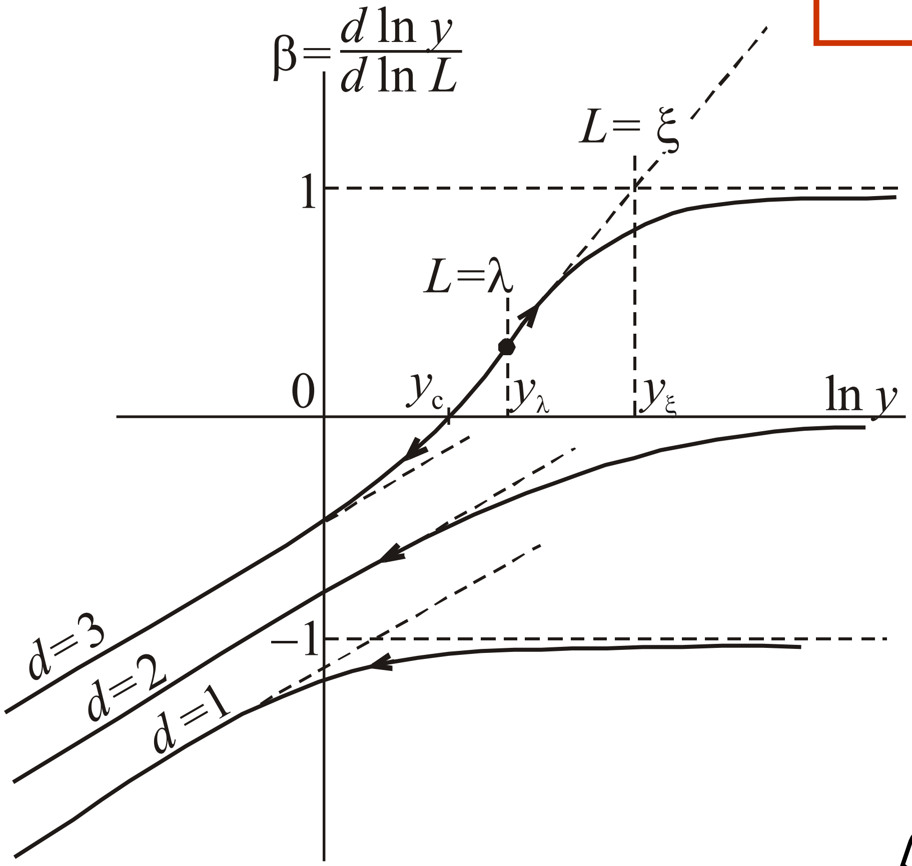
β - function

$$\frac{d\ln y}{d\ln L} = \beta(y)$$



Metal – insulator transition in 3D
All states are localized for d=1,2

$$Y = \sigma L^{d-2} \qquad y = \frac{Y}{e^2/\hbar} = \frac{\sigma L^{d-2}}{e^2/\hbar}$$



Two limits:

I. Metal σ = const

$$\ln y = \text{const} + (d-2) \ln L$$

$$\beta \equiv \frac{d \ln y}{d \ln L} \rightarrow d-2$$

II. Insulator $\sigma \longrightarrow 0$

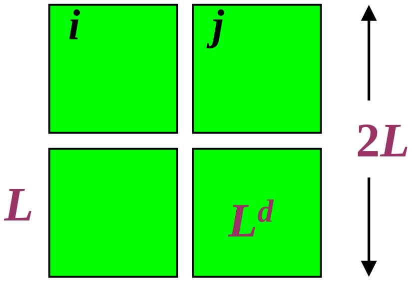
$$y = y_0 e^{-L/a_B}$$

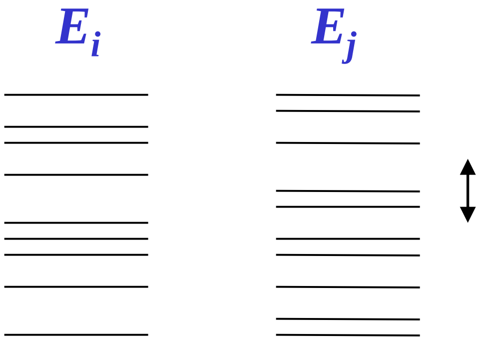
$$\beta \equiv \frac{d \ln y}{d \ln L} = L \frac{d}{dL} \ln y = -L/a_B = \ln y / y_0$$

E.Abrahams, P.W.Anderson, D.C.Licciardello, and T.W.Ramakrishnan, Phys.Rev.Lett. 42, 673 (1979)

Physical background of the scaling hypothesis:

matching 2^d hypercubes volume L^d each to one hypercube volume $(2L)^d$





$$\overline{\delta E} \sim (g_d L^d)^{-1}$$

Perturbation theory

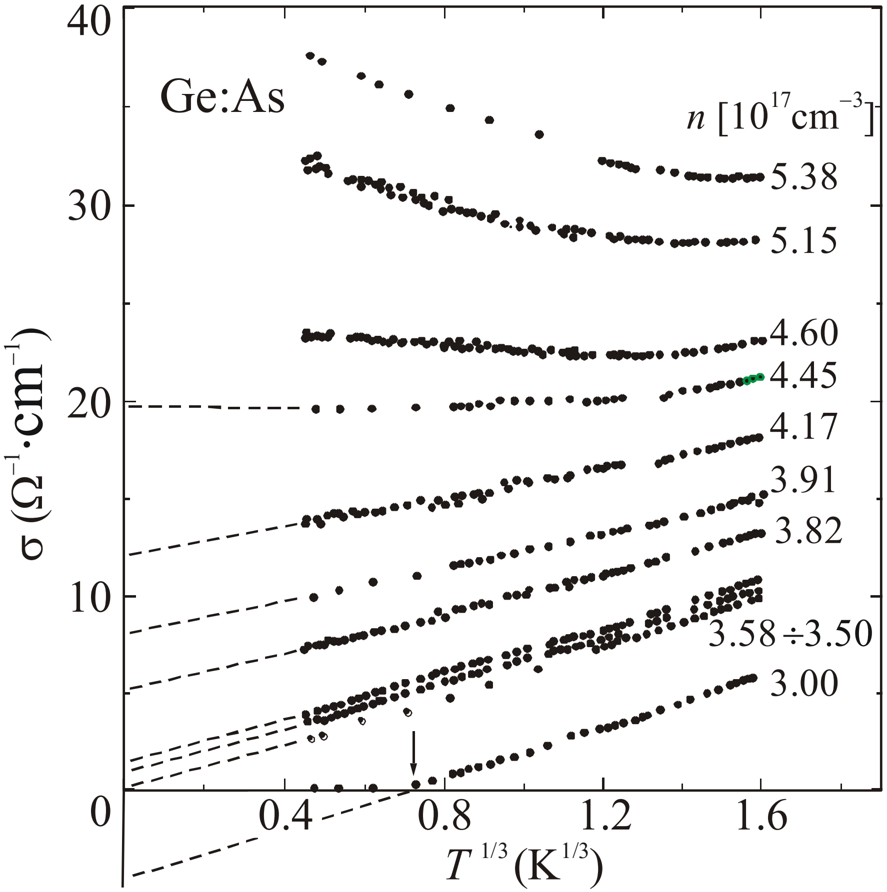
$$\psi_i \sim \psi_{i0} + c_{ij}\psi_{j0}$$

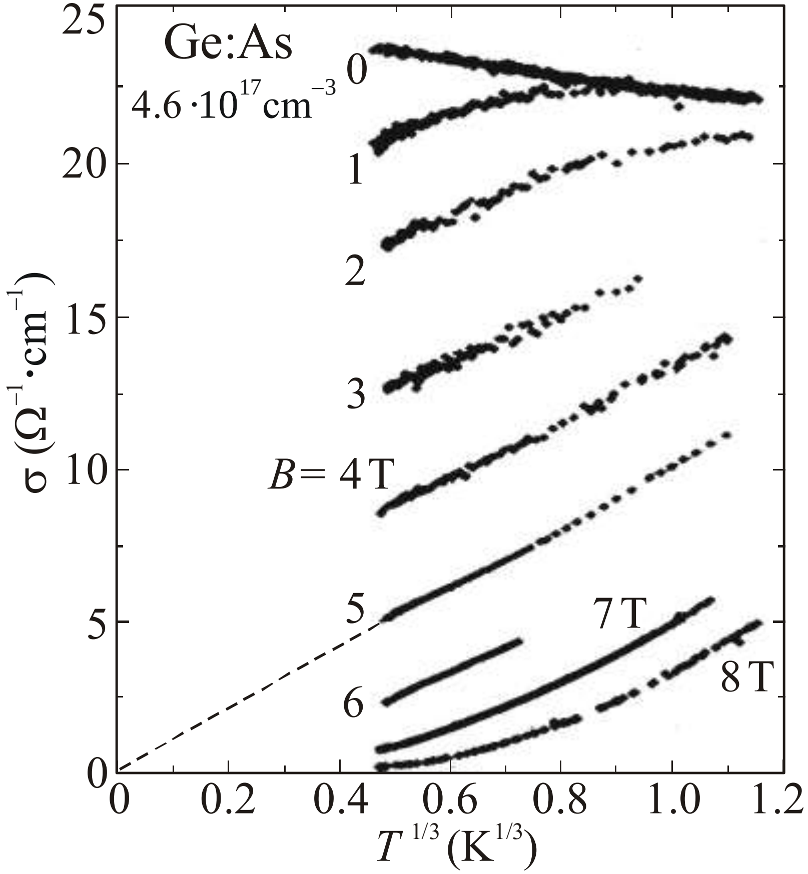
$$c_{ij} \sim \frac{J}{E_i - E_j} \sim J(g_d L^d) \sim \frac{J}{W} n_d L^d$$

This ratio enters Anderson localization criterium

with changing size L changes of wave functions depend on overlap integral J , similar to conductance

***d* = 3** (*experiment*)

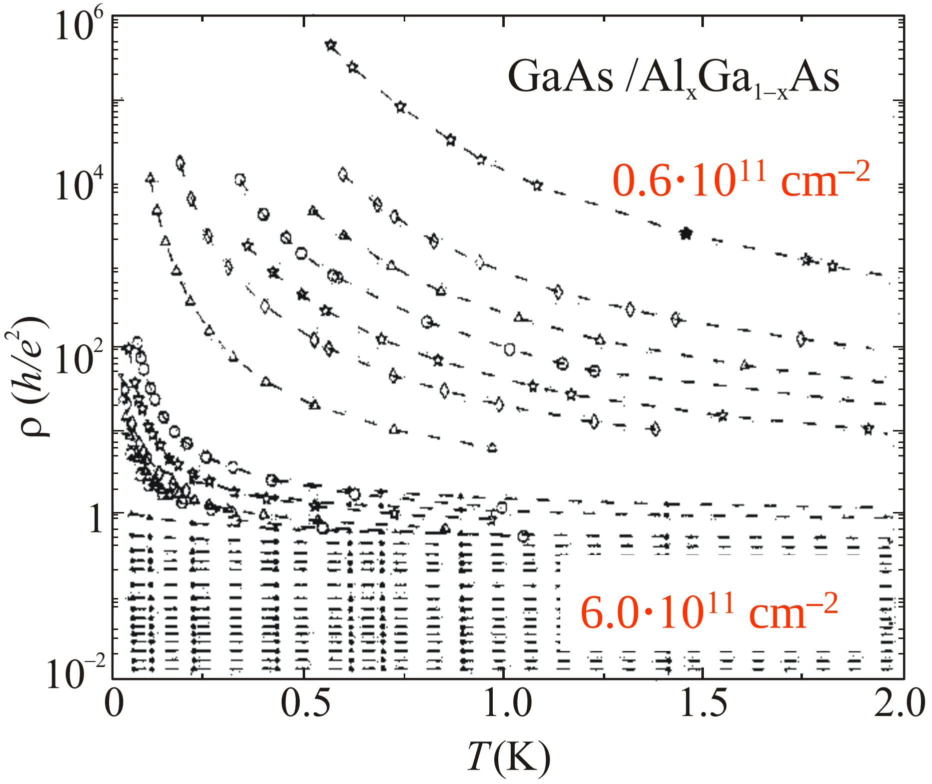




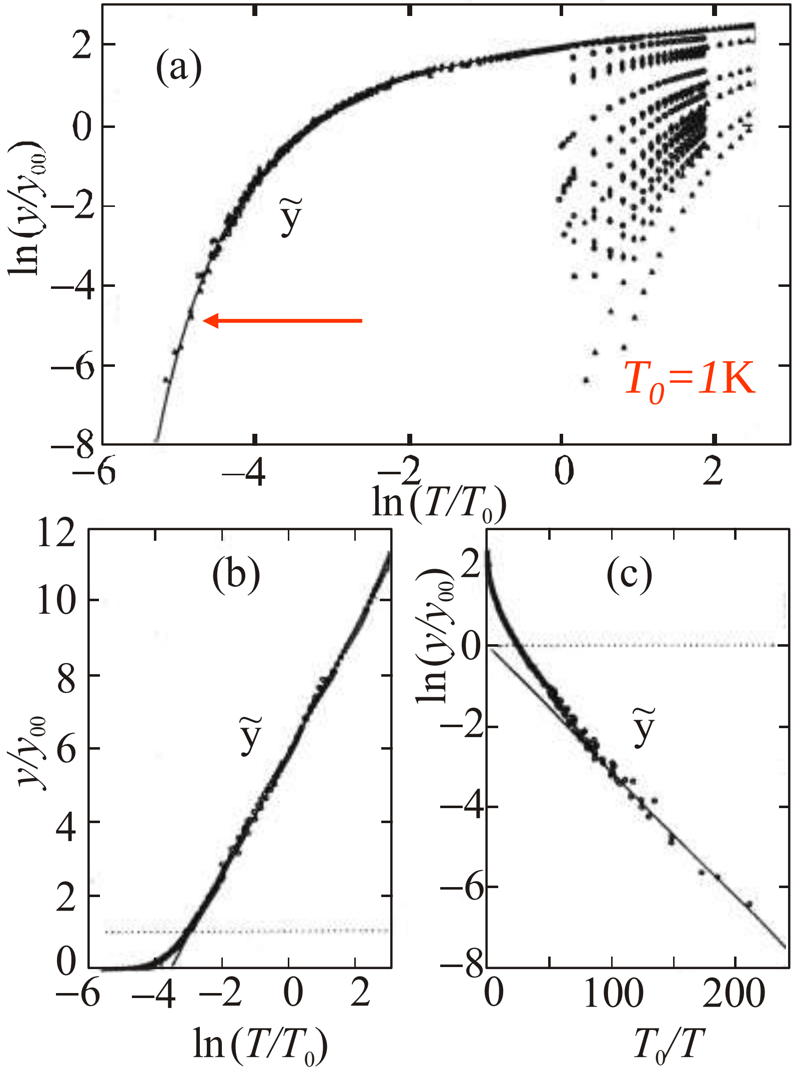
I.Shlimak, M.Kaveh, R.Ussyshkin, et al., Phys.Rev.Lett. 77,1103 (1996)

I.Shlimak, M.Kaveh, R.Ussyshkin, et al., Phys.Rev.B 55,1303 (1997)

***d* = 2 (*experiment*)**



F.W. van Keuls et al., Phys.Rev. B 56,13263 (1997)

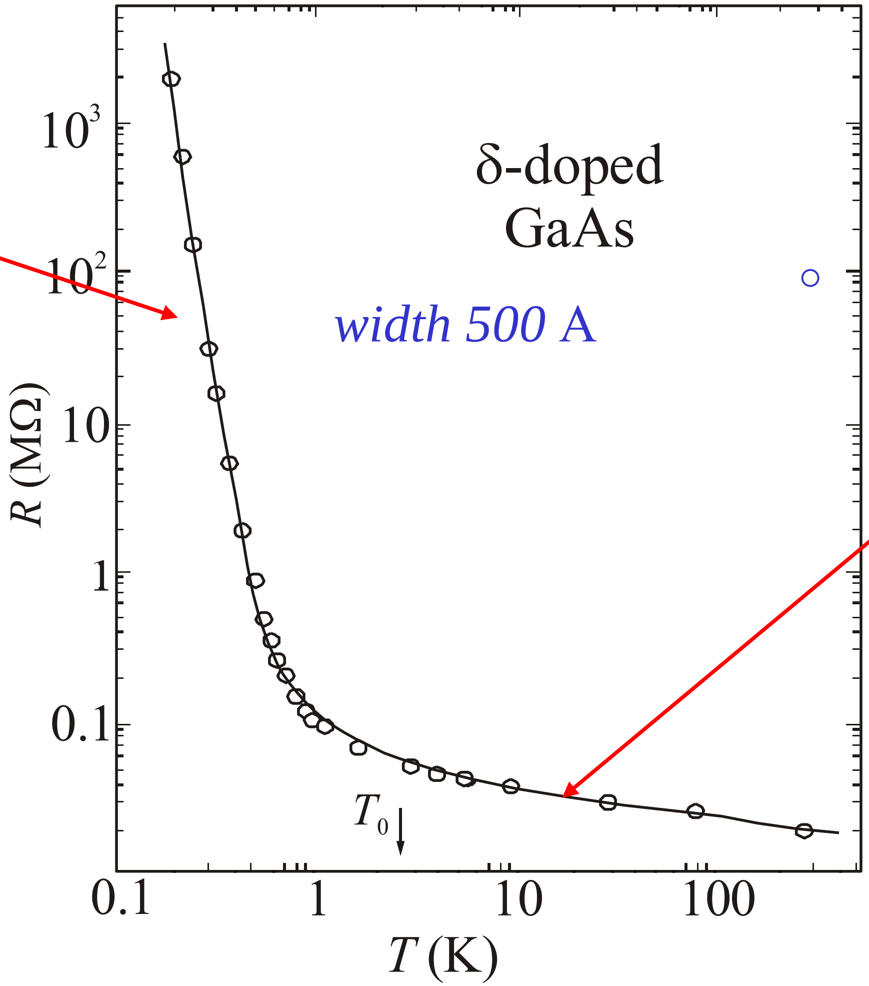


Films Cu/Ge, Ag/Ge , Au/Ge 0.3-20 Å

S.-Y.Hsu and J.M.Valles, Jr., Phys.Rev.Lett. 74,2331 (1995)

d = 2 (*experiment*)

Strong localization

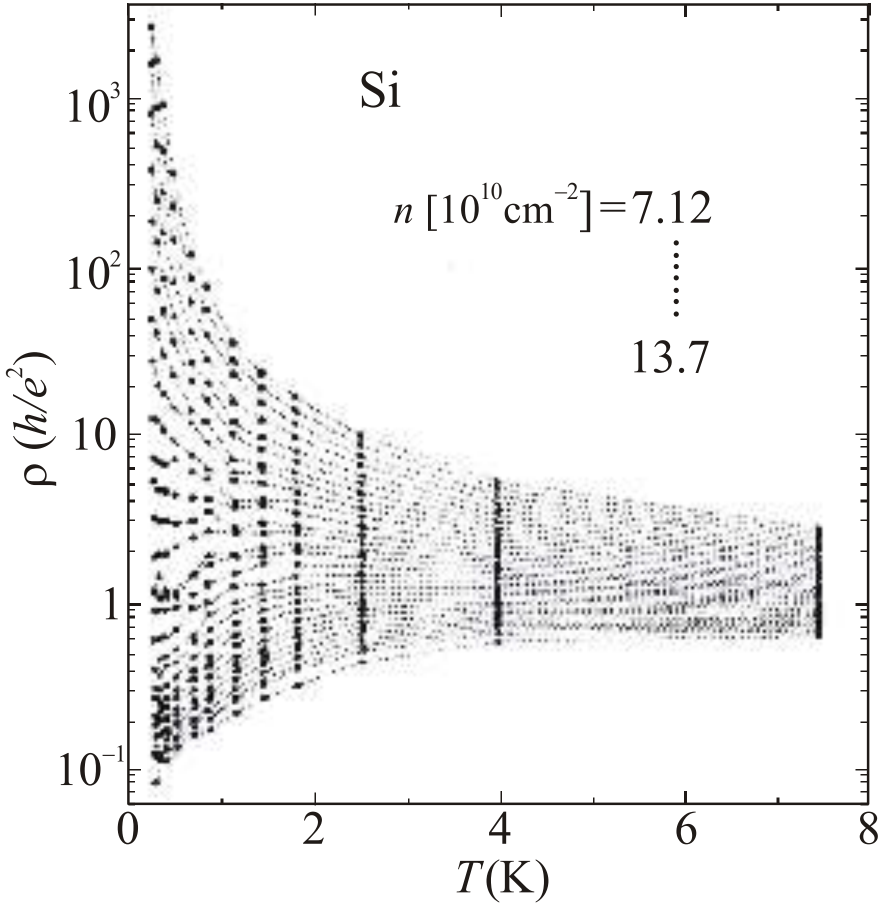


Weak localization

Yu.Havin,M.Gershenson, and A.Bogdanov,
Phys.Rev. B 58,8009 (1998)

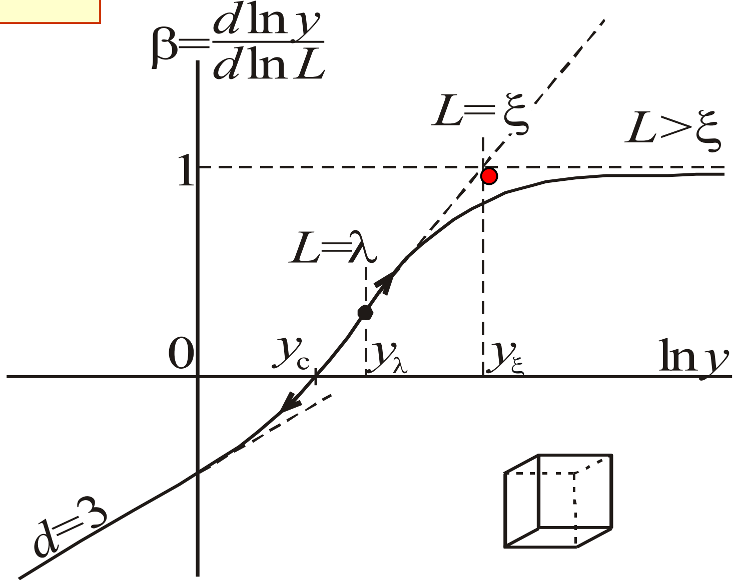
d **= 2** *(experiment)*

2DEG on Si surface



S.V.Kravchenko, W.E.Mason, G.E.Bowker, et al., Phys.Rev. B 51,7038 (1995)

$d = 3$



Near the transition point

$$\frac{d\ln y}{d\ln L} = s\ln {}^{y}\!/_{y_c}$$

$u=\ln y,\quad x=\ln L$

$$\frac{du}{dx} = s(u-u_c)$$

$$u-u_c = Ue^{sx}$$

The solution

$$\ln {}^{y}\!/_{y_c} = \left(\frac{L}{\lambda}\right)^{s} \ln {}^{y_\lambda}\!/_{y_c}$$

at point ● $\beta=1$ *and* $\ln y_\xi/y_c = 1/s$ =>

$$\ln y_\xi = \ln y_c + {}^{1}\!/_{s} = \text{const} \Longrightarrow y_\xi = A \Longrightarrow Y_\xi = A\frac{e^2}{\hbar} \Longrightarrow \boxed{\sigma = \left(A\frac{e^2}{\hbar}\right)\frac{1}{\xi}}$$

quantity ξ can be expressed via λ

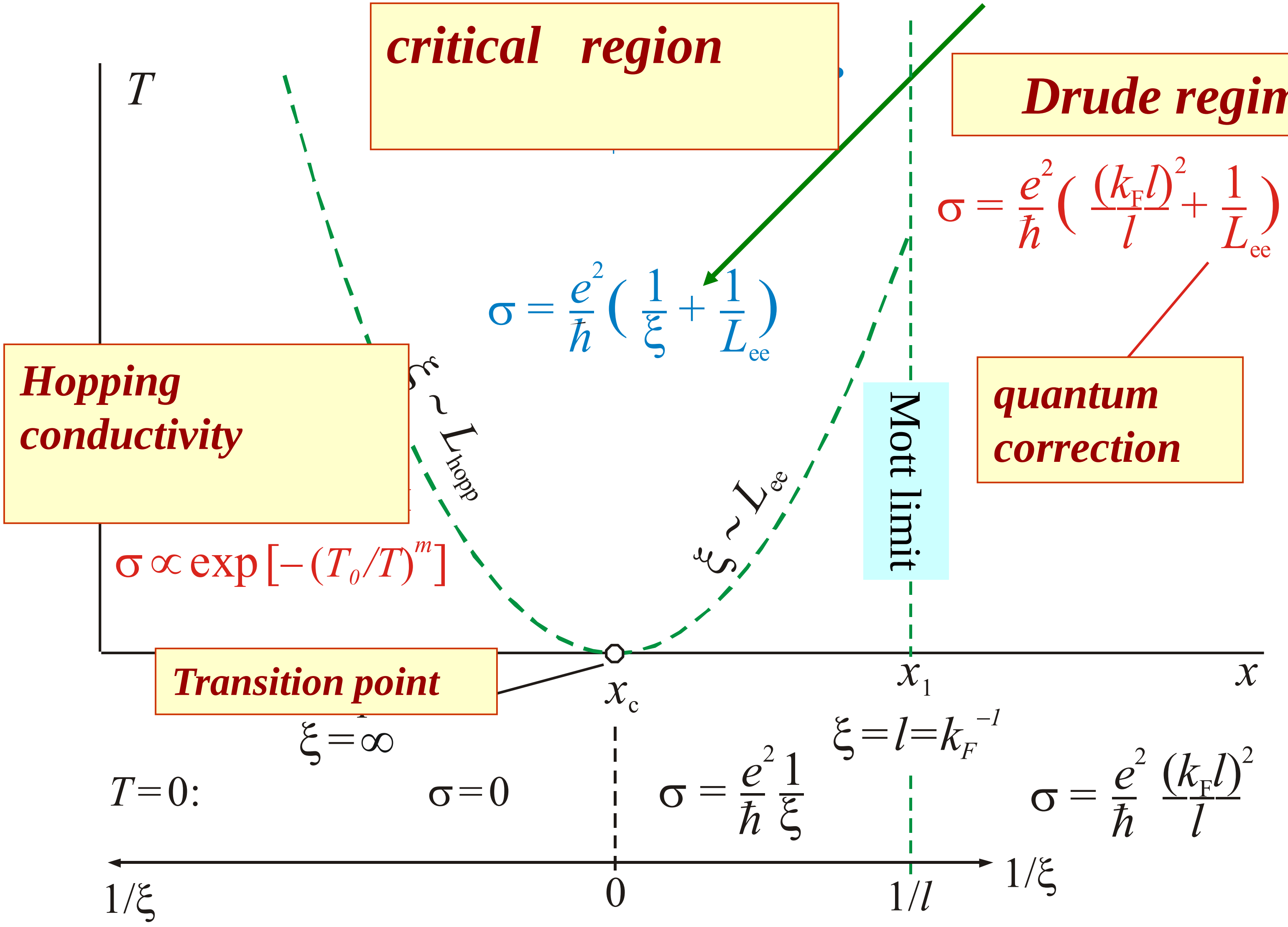
$$\xi \approx \lambda\left(s\ln\frac{y_\lambda}{y_c}\right)^{-{}^{1}\!/_{s}},$$ *i.e.*

$$\xi \to \infty$$
at $y_\lambda \to y_c$

conductivity σ
can become small

$d = 3$

Two length scales: ξ and L_φ



critical region

$$\sigma = \frac{e^2}{\hbar}\left(\frac{1}{\xi} + \frac{1}{L_{\text{ee}}}\right)$$

Drude regime

$$\sigma = \frac{e^2}{\hbar}\left(\frac{(k_{\text{F}}l)^2}{l} + \frac{1}{L_{\text{ee}}}\right)$$

quantum correction

Hopping conductivity

$$\sigma \propto \exp[-(T_0/T)^m]$$

Transition point

Mott limit

$T = 0$:

$\xi = \infty$

$\sigma = 0$

$$\sigma = \frac{e^2}{\hbar}\frac{1}{\xi}$$

$\xi = l = k_F^{-1}$

$$\sigma = \frac{e^2}{\hbar}\frac{(k_{\text{F}}l)^2}{l}$$

$d = 3$

$\sigma(T)$ in the critical region

Metallic region at $T = 0$

$$\sigma = \sigma_{03} + \frac{e^2}{\hbar}\frac{1}{L_T} \qquad L_T = \sqrt{D\hbar/T},$$

Near transition point at $T = 0$

$$\sigma = \frac{e^2}{\hbar}\frac{1}{\xi}$$

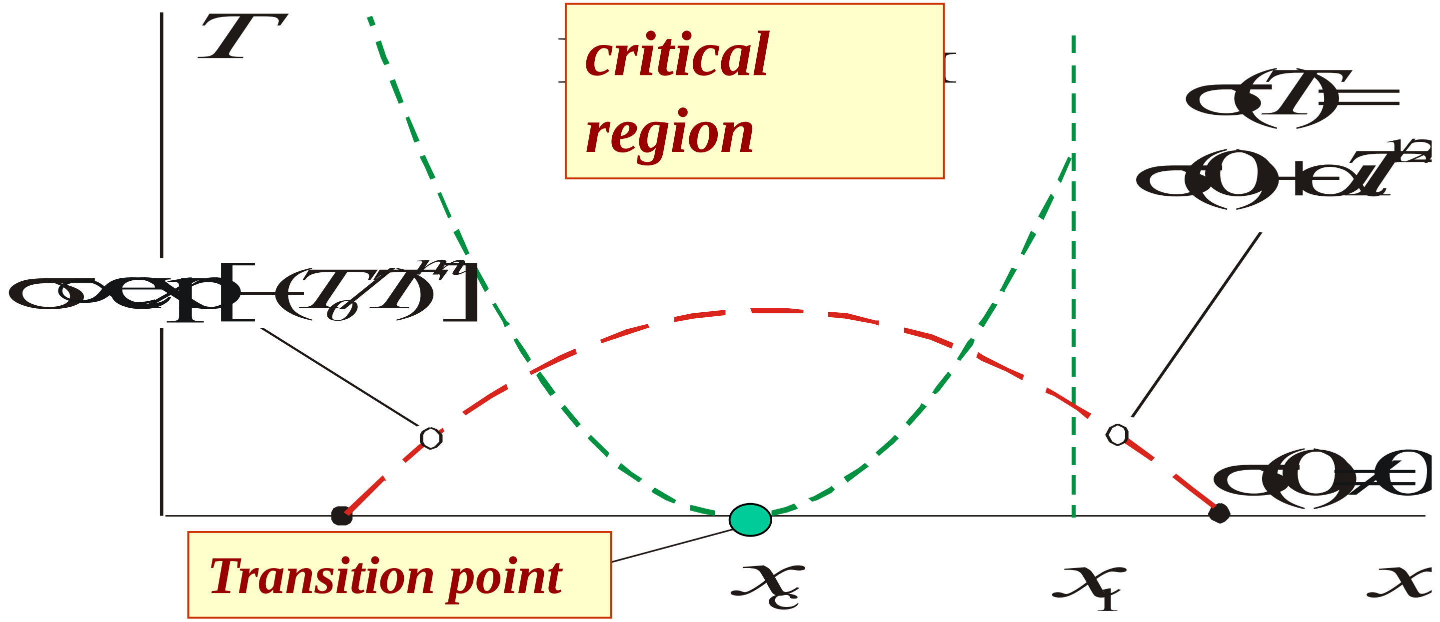
Interpolation formula

$$\boxed{\sigma = \frac{e^2}{\hbar}\left(\frac{1}{\xi} + \frac{1}{L_T}\right)}$$

$$\left\{\begin{aligned} \sigma &= \frac{e^2}{\hbar}\left(\frac{1}{\xi} + \frac{1}{L_T}\right) \\ \sigma &= e^2 g_F D \end{aligned}\right. \quad \xrightarrow[\xi \to \infty]{} \quad \boxed{\sigma = \frac{e^2}{\hbar}(T g_F)^{1/3}}$$

d = 3

(x,T) diagram



$$\sigma = \frac{e^2}{\hbar}\left(\frac{1}{\xi} + \frac{1}{L_T}\right)$$

2D

$$\left.\begin{aligned}\sigma &= \sigma_{02} - \frac{e^2}{\hbar}\ln\frac{L_T}{l} \\ \sigma_{02} &= \frac{ne^2 l}{\hbar k_F} \approx \frac{e^2}{\hbar}(k_F l)\end{aligned}\right\}\quad k_F l = \ln L_T/l$$

$$n \propto k_F^2$$

$$L_T \equiv \xi = l\exp(k_F l)$$

$$L_T = \sqrt{D\hbar/T},$$

$$\boxed{T_\xi = D\hbar/\xi^2 \approx \frac{\hbar}{\tau}\exp[-2(k_F l)]}$$

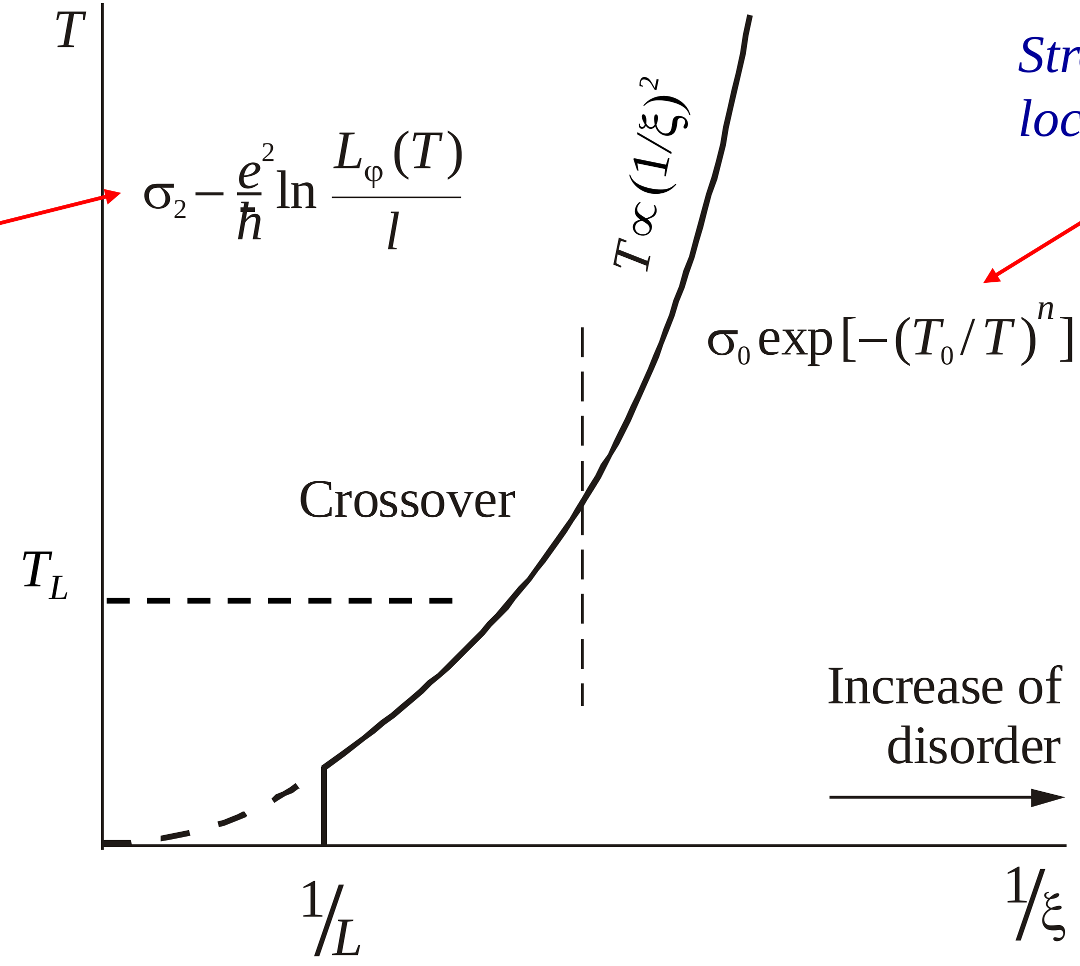
Even not too large values $k_F l$ ~ 10

provide extremely low T_ξ

2D regime: crossover in T-dependence instead of phase transition at T=0

Weak

localization



$$\sigma_2 - \frac{e^2}{\hbar}\ln\frac{L_\varphi(T)}{l}$$

$T \propto (1/\xi)^2$

Strong localization

$$\sigma_0 \exp[-(T_0/T)^n]$$

Crossover

Increase of disorder

$$T_L \approx \frac{\hbar}{\tau}\left(\frac{l}{L}\right)^2$$

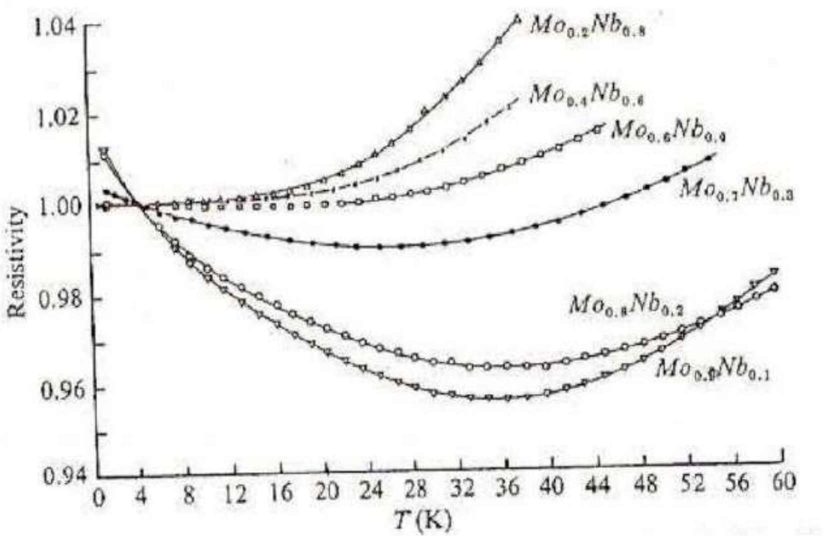
Thouless energy,

might be not small

$$\tau_\varphi = \varepsilon_F \tau / T, \quad L_\varphi = \sqrt{D\tau_\varphi} = \sqrt{D\varepsilon_F \tau / T}$$

$$L_\varphi = \xi \quad \Rightarrow \quad T_{cr} = \frac{D\varepsilon_F \tau}{\xi^2} = \varepsilon_F \exp(-2k_F l)$$

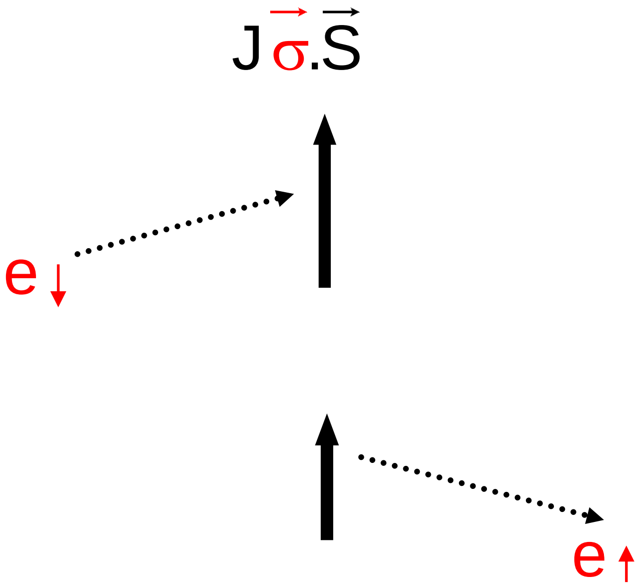
Kondo effect



There is resistance minimum in some cases when impurity is magnetic

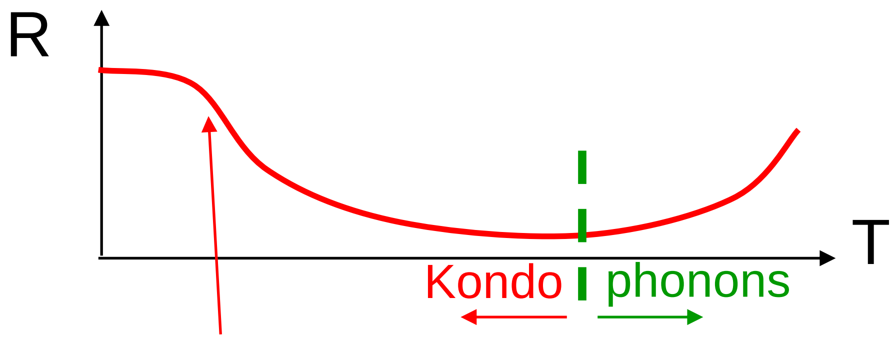
Theoretical explanation:
Jun Kondo (1964)

Magnetic impurities: the origin of the Kondo effect



Spin-flip scattering

If J < 0, resistivity increases at low T:



T_K - ***Kondo temperature***

$$\rho(T)=\rho_0(T)+\rho_1\left(1-2J\nu\ln\frac{E_F}{k_BT}\right)$$

ν - density of states

J - exchange coupling: ***J > 0*** – ferromagnetic, ***J < 0*** – antiferromagnetic

Nature of spin-spin coupling:
Exchange interaction

Consider two electrons in states with orbital wave functions $\varphi_1(\mathbf{r})$ and $\varphi_2(\mathbf{r})$

Total wave function $$\varphi = \frac{1}{\sqrt{2}}\left[\varphi_1(\mathbf{r}_1)\,\varphi_2(\mathbf{r}_2) \pm \varphi_1(\mathbf{r}_2)\,\varphi_2(\mathbf{r}_1)\right]$$

sign ‘+’ for total spin S = 0, sign ‘-’ for spin S = 1 (since Orbital+Spin = Even)

Average values of interaction energy $U(\mathbf{r}_2 - \mathbf{r}_1)$ are $A \pm J$

$$A = \iint U\,|\varphi_1(\mathbf{r}_1)|^2\,|\varphi_2(\mathbf{r}_2)|^2\,dV_1 dV_2,$$
$$J = \iint U\varphi_1(\mathbf{r}_1)\,\varphi_1^*(\mathbf{r}_2)\,\varphi_2(\mathbf{r}_2)\,\varphi_2^*(\mathbf{r}_1)\,dV_1 dV_2$$

⇒ the shifts of energy levels $\Delta E_0 = J, \quad \Delta E_1 = -J$

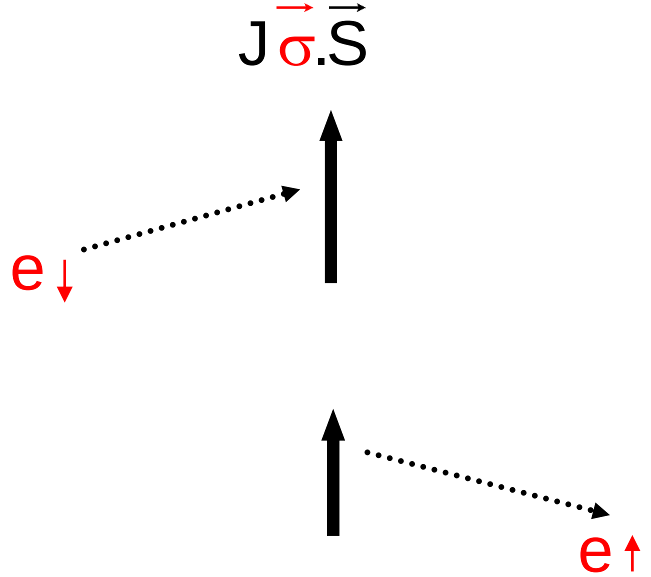
J - **“exchange energy”** (***J>0 – ferromagnetic, J<0 - antiferromagnetic***)

energy levels are *eigenvalues* of the exchange operator $$-\frac{1}{2}J(1+4\vec{\sigma}_1\vec{\sigma}_2)$$

$\vec{\sigma}_{1,2}$ - operators of electron spin

Interaction of electron spin $\vec{\sigma}$ with impurity spin $\vec{S}$:

the exchange operator has the form $-J\vec{\sigma}\vec{S}$



Correction to resistivity: scattering rate is calculated in the second order of perturbation theory.

For derivation see the textbook of Abrikosov, Chapter 4.6

Initial state of an electron is (p,σ), final state is $(p'\ \sigma')$
Second order perturbation theory:
scattering occurs via intermediate state (p_1,σ_1)

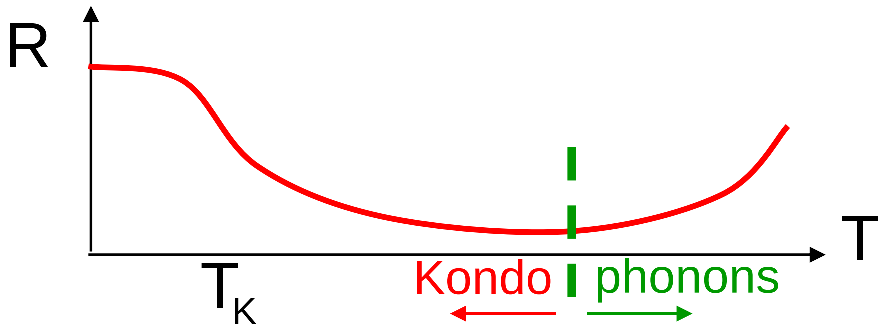
(1) $\sigma \to \sigma_1$, than $\sigma_1 \to \sigma'$. The state $(p_1\sigma_1)$ should be empty,
= > factor $[1- f(p_1)]$, where $f(p_1)]$ is the Fermi function

$$\left(\frac{J}{n}\right)^2 \sum_{\sigma_1} \int \frac{(\sigma S)_{\sigma'\sigma_1}(\sigma S)_{\sigma_1\sigma}(1-f(p_1))}{\varepsilon(p)-\varepsilon(p_1)} \frac{d^3p_1}{(2\pi\hbar)^3}$$

2) $\sigma_1 \to \sigma'$, than $\sigma \to \sigma_1$
The state $(p_1\sigma_1)$ should be occupied = > factor $f(p_1)$

$$-\left(\frac{J}{n}\right)^2 \sum_{\sigma_1} \int \frac{(\sigma S)_{\sigma_1\sigma}(\sigma S)_{\sigma'\sigma_1} f(p_1)}{\varepsilon(p_1)-\varepsilon(p')} \frac{d^3p_1}{(2\pi\hbar)^3}$$

The origin of resistance increase at T - > 0



$$\rho(T) = \rho_0(T) + \rho_1\left(1 - 2J\nu \ln\frac{E_F}{k_B T}\right)$$

If J < 0 resistance increases for T -> 0

Physical reason : for J < 0 tendency to antiparallel spin orientations, S_z= S.
Total spin of electron and impurity is S - ½
Initial spin state is (-1/2, S), final state is (1/2, S -1)

If J > 0, total spin of electron and impurity is S + ½
if electron scatters from spin state ½ to - ½, impurity spin should become S + 1 – not possible, therefore scattering is suppressed for J > 0

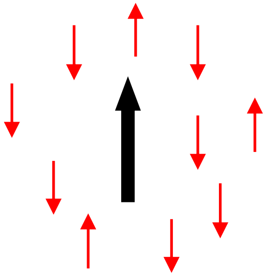
Resistivity saturation at below Kondo temperature

$$\rho(T) = \rho_0(T) + \rho_1\left(1 - 2J\nu\ln\frac{E_F}{k_B T}\right)$$

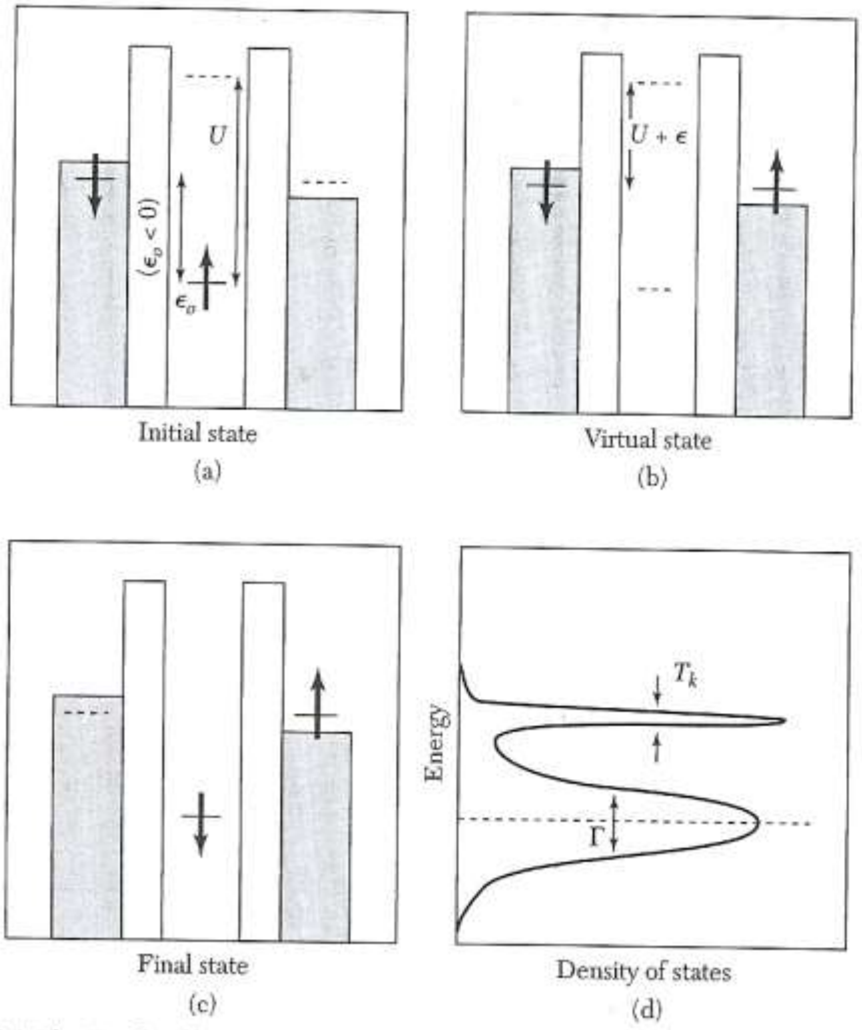
Kondo temperature $$k_B T_K \approx E_F e^{-\frac{1}{\nu|J|}}$$

Below Kondo temperature impurity spin is screened by other electrons => resistance saturation

collective effect



Kondo effect in a quantum dot



Initial state
(a)

Virtual state
(b)

Final state
(c)

Density of states
(d)

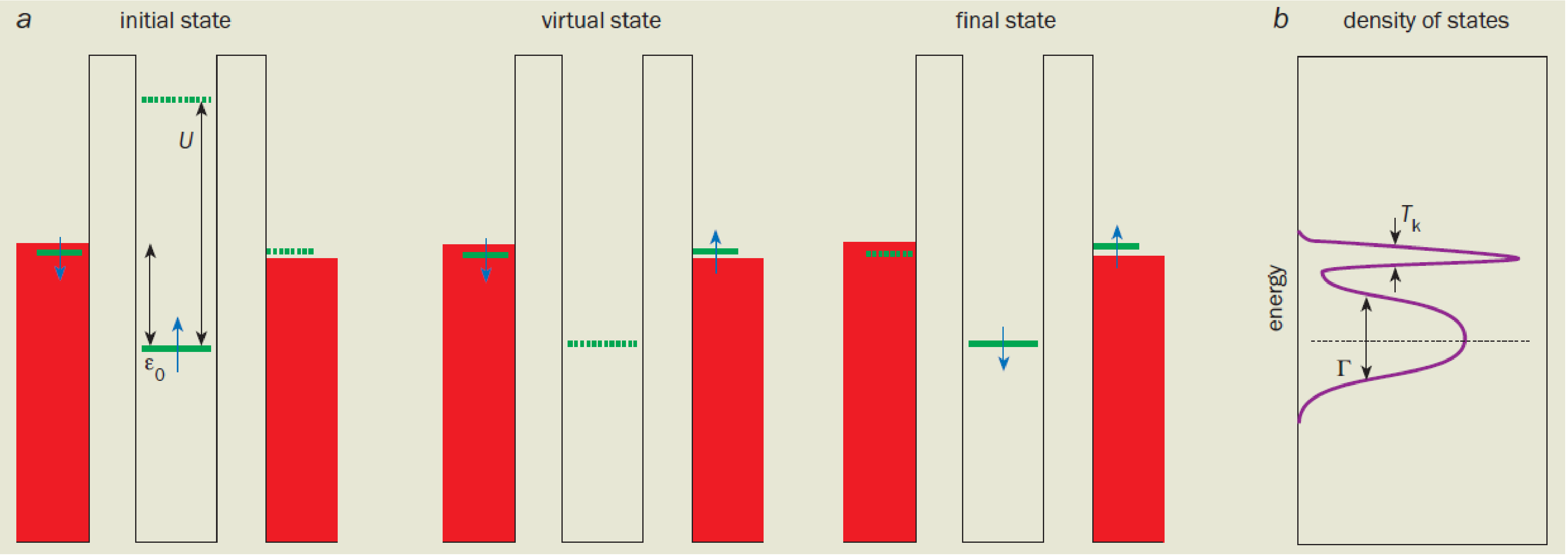
Figure 24 The Kondo effect in a quantum dot. For an unpaired spin on the dot, a virtual process (b) can occur that converts the spin up (a) to the spin down (c) state and transfers an electron from one side of the dot to the other. The ground state of the system is a coherent superposition on the initial and final states shown, creating a spin singlet between the spin on the dot and the spins in the leads. This is called the Kondo effect, and produces a narrow peak of width $\sim k_B T_K$ in the density of states at ε_F in addition to the original broadened level of width Γ, as shown in (d).

From C.Kittel
'Introduction to
Solid State Physics'

Chapter 18
'Nanostructures'

Analogy to the Mott insulator

$$k_B T_K = \frac{1}{2}(\Gamma U)^{1/2}\exp[\frac{\pi\varepsilon_0(\varepsilon_0 + U)}{\Gamma U}]$$



L. Kouwenhoven *et al.*, Phys. World **14, 33 (2001).**