BIT-MIPT Summer School

Microscopic theory of mesoscopic superconductivity

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International summer school at Beijing Institute of Technology, 16th – 26th July 2023

Outline of mini-course

- ❖ Microscopic BCS (Bardin-Cooper-Schriffer) theory of superconductivity
- ❖ Formalism of Bogolubov- de Gennes equations
- ❖ Andreev reflection, Andreev bound states
- ❖ NIS interface, Blonder-Tinkham-Klapwijk Formalism

https://mezo-mipt.ru/schools/5/en

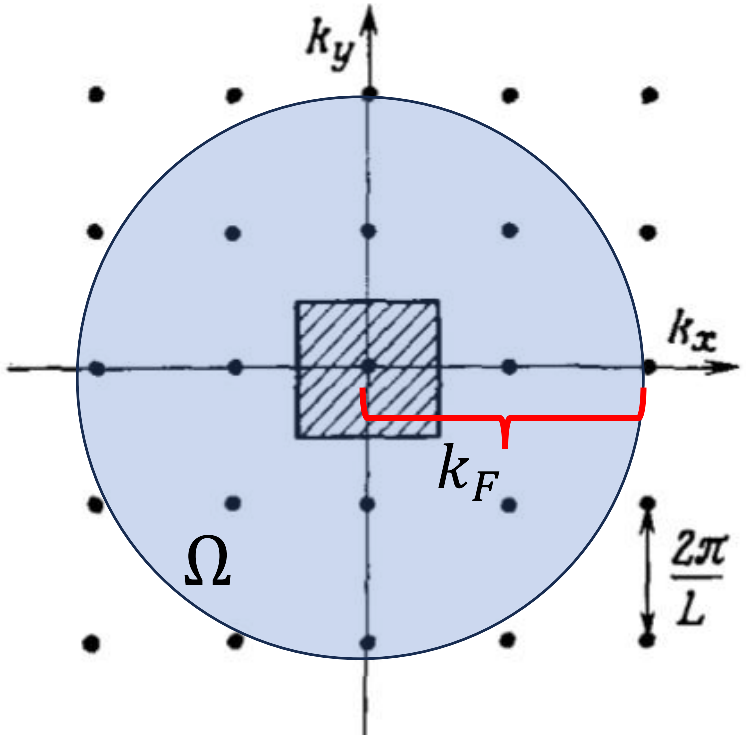
Requirements

- Differential and integral calculus, including calculus of variations
- Basics of quantum mechanics including the second quantization formalism
- Basics of solid-state physics
- Basics of thermodynamics
- Basics of electrodynamics

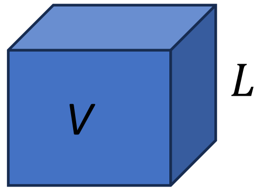
Outline of the 1st section – theoretical background

- Landau Fermi liquid
- The Cooper problem
- The BCS model
- The Bogolubov-de Gennes equations
- The self-consistency equation
- Observables

Landau Fermi liquid: ground state



Wave function of a free particle in a box



$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\boldsymbol{r}}$$

Periodic boundary conditions

$$\psi(x,y,z+L) = \psi(x,y,z)$$
$$\psi(x,y+L,z) = \psi(x,y,z)$$
$$\psi(x+L,y,z) = \psi(x,y,z)$$

$$e^{ik_xL} = e^{ik_yL} = e^{ik_zL} = 1 \Rightarrow$$
$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L}$$

Number of states in real volume *V* and k-space volume Ω

$$N_{k,V} = \frac{\Omega}{(2\pi/L)^3} = \frac{\Omega V}{(2\pi)^3}$$

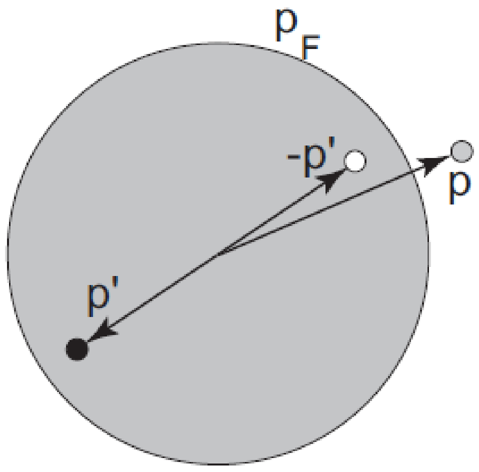
Maximal energy of electrons in the ground state

$$E_F$$

Fermi momentum

$$N = \frac{V}{(2\pi)^3}\frac{4\pi k_F^3}{3} \Rightarrow \boxed{n = \frac{k_F^3}{3\pi^2}}$$

Landau Fermi liquid: quasiparticles



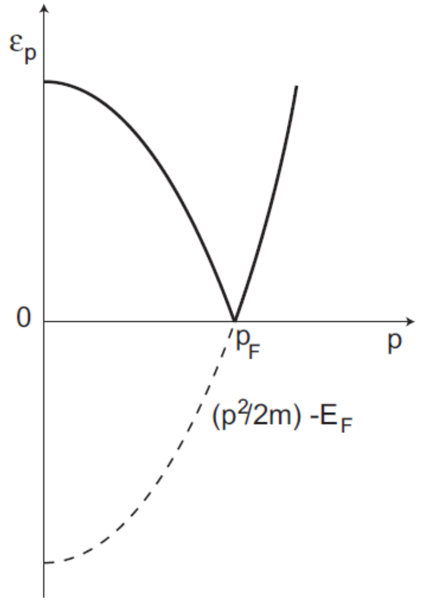
Two-stage process of creation an excitation:

I. A removal of a particle out of the state below E_F.
   Result: a hole excitation with $\epsilon_1 = E_F - E_1$

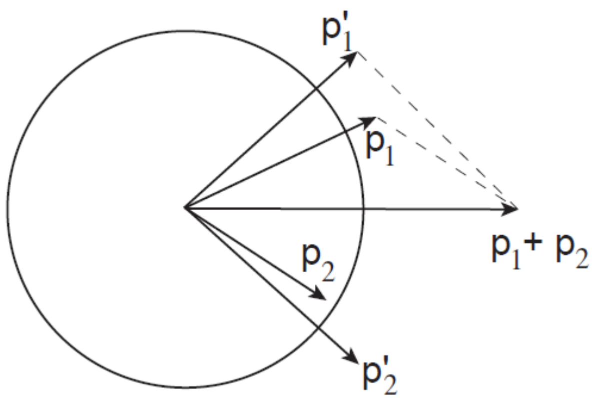
II. Adding a particle to a state above E_F
   Result: a particle excitation with $\epsilon_2 = E_2 - E_F$

Thus, for isotropic system the quasiparticle spectrum is:

$$\epsilon_{\mathbf{p}} = \begin{cases} \frac{p^2}{2m} - E_F, & p > p_F \\ E_F - \frac{p^2}{2m}, & p < p_F \end{cases}$$



Landau Fermi liquid: quasiparticle lifetime



Scattering of quasiparticles

Pauli principle: $p_1 > p_F \,,\; p_2 < p_F \,,\; p_1' > p_F \,,\; p_2' > p_F$

Momentum conservation: $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$

The probability of the scattering process:

$$\begin{aligned} P &= 2m \int M\delta\left(p_1^2 + p_2^2 - p_1'^2 - p_2'^2\right)\delta\left(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2'\right) d^n p_2\, d^n p_1'\, d^n p_2' \\ &= 2m \int M\delta\left[p_1^2 + p_2^2 - p_1'^2 - \left(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1'\right)^2\right] d^n p_2\, d^n p_1' \end{aligned}$$

All the momenta are close to the Fermi surface $\Longrightarrow \boldsymbol{p}_1 \approx \boldsymbol{p}_1', \boldsymbol{p}_2 \approx \boldsymbol{p}_2' \Longrightarrow$

$$p_1 + p_2 = p_1' + p_2'$$

Since $p_2' > p_F$, we have

$$p_1' = p_1 + p_2 - p_2' < p_1 + p_2 - p_F$$

Therefore,

$$0 < p_1' - p_F < (p_1 - p_F) + (p_2 - p_F)$$

On the other hand,

$$p_F - p_2 < p_1 - p_1' < p_1 - p_F$$

thus

$$0 < p_F - p_2 < p_1 - p_F$$

Landau Fermi liquid: quasiparticle lifetime

$$
\begin{aligned}
P &\propto \int dp_2\, dp_1' = \int_0^{p_1-p_F} d(p_F-p_2) \int_0^{(p_1-p_F)+(p_2-p_F)} d(p_1'-p_F) \\
&= \int_0^{p_1-p_F} \left[(p_1-p_F)-(p_F-p_2)\right] d(p_F-p_2) = \frac{1}{2}\,(p_1-p_F)^2
\end{aligned}
$$

The coefficient of proportionality can be established from the dimensions

$$
P \sim \frac{v_F^2\,(p_1-p_F)^2}{\hbar E_F} = \frac{\epsilon^2}{\hbar E_F}
$$

This means that uncertainty of the quasiparticle energy $\delta\epsilon \sim \hbar P$ is small compared to the energy if $\epsilon \ll E_F$, i.e., near the Fermi surface. In other words, quasiparticles are well defined only near the Fermi surface.

Landau Fermi liquid: Hamiltonians for particles and holes

One-particle Hamiltonian for particles:

$$\hat{H}_e = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 + U_0(\mathbf{r}) - \mu$$

$$\hat{H}_e u_{\epsilon,\mathbf{p}}(\mathbf{r}) = \epsilon_{\mathbf{p}} u_{\epsilon,\mathbf{p}}(\mathbf{r})$$

What is the one-particle Hamiltonian for holes?

$$\hat{H}_h v_{\epsilon,\mathbf{p}}(\mathbf{r}) = \epsilon_{\mathbf{p}} v_{\epsilon,\mathbf{p}}(\mathbf{r})$$

$$v_{\epsilon,\boldsymbol{p}} = u_{-\epsilon,-\boldsymbol{p}}$$

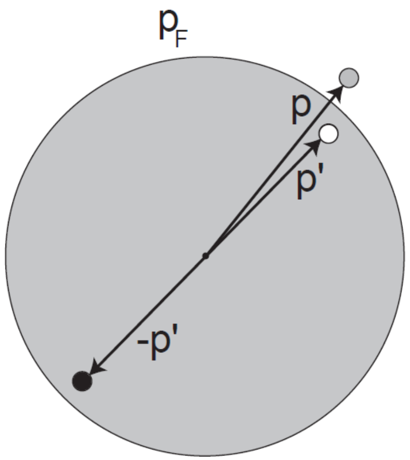
$$\hat{H}_e(-\boldsymbol{p}) = \hat{H}_e^*$$

$$\hat{H}_e^* = \frac{1}{2m}\left(i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 + U_0(\mathbf{r}) - E_F$$

$$\hat{H}_e^* v_{\epsilon,\mathbf{p}}(\mathbf{r}) = -\epsilon_{\mathbf{p}} v_{\epsilon,\mathbf{p}}(\mathbf{r})$$

$$\boxed{\hat{H}_h = -\hat{H}_e^*}$$

The Cooper problem



Assume pairing correlations between electrons $\boldsymbol{p}$ and $-\boldsymbol{p}' \approx -\boldsymbol{p}$

Annihilating the pair we annihilate electron $\boldsymbol{p}$ and create hole $\boldsymbol{p}'$ ⇒

correlations between them should exist

$$u_{\mathbf{p}}(\mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}U_{\mathbf{p}} \qquad v^*_{\mathbf{p}'}(\mathbf{r}) = e^{-i\mathbf{p}'\cdot\mathbf{r}/\hbar}V^*_{\mathbf{p}}$$

The pair wave function is

$$\Psi^{\text{pair}}_{\mathbf{p}}(\mathbf{r}_1,\mathbf{r}_2) = u_{\mathbf{p}}(\mathbf{r}_1)v^*_{\mathbf{p}}(\mathbf{r}_2) = e^{i\mathbf{p}\cdot(\mathbf{r}_1-\mathbf{r}_2)/\hbar}U_{\mathbf{p}}V^*_{\mathbf{p}}$$

The linear combination with various $\mathbf{p}$ gives the coordinate wave function

$$\Psi^{\text{pair}}(\mathbf{r}_1,\mathbf{r}_2) = \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{r}_1-\mathbf{r}_2)/\hbar}a_{\mathbf{p}}$$

$$a_{\mathbf{p}} = U_{\mathbf{p}}V^*_{\mathbf{p}}$$

The Cooper problem

$$\left[\hat{H}_e(\mathbf{r}_1) + \hat{H}_h(\mathbf{r}_2) + W(\mathbf{r}_1, \mathbf{r}_2)\right] \Psi^{\text{pair}}(\mathbf{r}_1, \mathbf{r}_2) = E\Psi^{\text{pair}}(\mathbf{r}_1, \mathbf{r}_2)$$

or, in the momentum representation,

$$[2\epsilon_{\mathbf{p}} - E_{\mathbf{p}}]\, a_{\mathbf{p}} = -\sum_{\mathbf{p}_1} W_{\mathbf{p},\mathbf{p}_1} a_{\mathbf{p}_1}$$

where

$$W_{\mathbf{p},\mathbf{p}_1} = \int e^{-i(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{r}/\hbar} W(\mathbf{r})\, d^3r$$

Assume that

$$W_{\mathbf{p},\mathbf{p}_1} = \begin{cases} W, & \epsilon_{\mathbf{p}} \text{ and } \epsilon_{\mathbf{p}_1} < E_c \\ 0, & \epsilon_{\mathbf{p}} \text{ or } \epsilon_{\mathbf{p}_1} > E_c \end{cases}$$

where $E_c \ll E_F$. We have

$$a_{\mathbf{p}} = \frac{W}{E - 2\epsilon_{\mathbf{p}}} \sum_{\mathbf{p}_1} a_{\mathbf{p}_1}$$

The Cooper problem

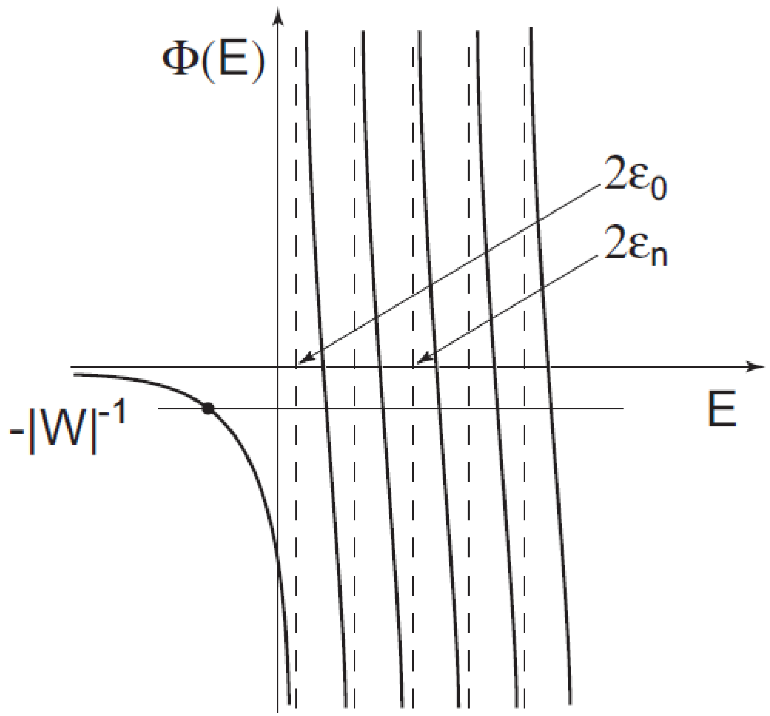
$$a_{\mathbf{p}} = \frac{W}{E - 2\epsilon_{\mathbf{p}}} \sum_{\mathbf{p}_1} a_{\mathbf{p}_1}$$

Let's solve this integral equation:

$$C = W \sum_{\mathbf{p}} \frac{C}{E - 2\epsilon_{\mathbf{p}}}$$

where $C = \sum_{\mathbf{p}} a_{\mathbf{p}}$; the sum is taken over $\mathbf{p}$ which satisfy $\epsilon_{\mathbf{p}} < E_c$. This gives

$$\frac{1}{W} = \sum_{\mathbf{p}} \frac{1}{E - 2\epsilon_{\mathbf{p}}} \equiv \Phi(E)$$



The Cooper problem

For an attraction $W<0$ we have

$$\frac{1}{|W|}=\sum_{\mathbf{p}}\frac{1}{2\epsilon_{\mathbf{p}}-E}$$

We substitute the sum with the integral

$$\sum_{\mathbf{p}}=\int\frac{d^3p}{(2\pi\hbar)^3}=\int\frac{mp}{2\pi^2\hbar^3}d\epsilon_{\mathbf{p}}$$

where $\epsilon_{\mathbf{p}}=p^2/2m$. As a result, for negative energy $E=-|E|$

$$N(0)=\frac{mp_F}{2\pi^2\hbar^3}$$

$$\frac{1}{|W|}=\int_0^{E_c}\frac{mp}{2\pi^2\hbar^3}\frac{d\epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}}-E}=N(0)\int_0^{E_c}\frac{d\epsilon}{2\epsilon+|E|}=\frac{N(0)}{2}\ln\left(\frac{|E|+2E_c}{|E|}\right)$$

$$|E|=\frac{2E_c}{e^{2/N(0)|W|}-1}$$

For weak coupling, $N(0)|W|\ll 1$, we find

$$|E|=2E_ce^{-2/N(0)|W|}$$

For a strong coupling, $N(0)|W|\gg 1$,

$$|E|=N(0)|W|E_c$$

The BCS model: Hamiltonian

Creation operators for electrons at point ***r*** with spins ↑,↓:

$$\left.\begin{aligned}\Psi^\dagger(\mathbf{r}\uparrow) &= \sum_n \left[\gamma^\dagger_{n\uparrow}u^*_n(\mathbf{r}) - \gamma_{n\downarrow}v_n(\mathbf{r})\right]\\ \Psi^\dagger(\mathbf{r}\downarrow) &= \sum_n \left[\gamma^\dagger_{n\downarrow}u^*_n(\mathbf{r}) + \gamma_{n\uparrow}v_n(\mathbf{r})\right]\end{aligned}\right\}$$

Bogolubov transformation

$$\begin{aligned}\Psi(\mathbf{r},\alpha)\Psi(\mathbf{r}',\beta) + \Psi(\mathbf{r}',\beta)\Psi(\mathbf{r},\alpha) &= 0\\ \Psi^\dagger(\mathbf{r},\alpha)\Psi^\dagger(\mathbf{r}',\beta) + \Psi^\dagger(\mathbf{r}',\beta)\Psi^\dagger(\mathbf{r},\alpha) &= 0\\ \Psi^\dagger(\mathbf{r},\alpha)\Psi(\mathbf{r}',\beta) + \Psi(\mathbf{r}',\beta)\Psi^\dagger(\mathbf{r},\alpha) &= \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')\end{aligned}$$

$$\begin{aligned}\gamma_{n,\alpha}\gamma_{m,\beta} + \gamma_{m,\beta}\gamma_{n,\alpha} &= 0\\ \gamma^\dagger_{n,\alpha}\gamma^\dagger_{m,\beta} + \gamma^\dagger_{m,\beta}\gamma^\dagger_{n,\alpha} &= 0\\ \gamma^\dagger_{n,\alpha}\gamma_{m,\beta} + \gamma_{m,\beta}\gamma^\dagger_{n,\alpha} &= \delta_{\alpha\beta}\delta_{mn}\end{aligned}$$

$\Rightarrow$

$$\sum_n [u^*_n(\mathbf{r})u_n(\mathbf{r}') + v^*_n(\mathbf{r}')v_n(\mathbf{r})] = \delta(\mathbf{r}-\mathbf{r}')$$

$$\sum_n [u^*_n(\mathbf{r})v_n(\mathbf{r}') - u^*_n(\mathbf{r}')v_n(\mathbf{r})] = 0$$

If the state is specified by a wave vector $\mathbf{q}$ such that $u_\mathbf{q}, v_\mathbf{q} \propto e^{i\mathbf{q}\cdot\mathbf{r}}$, we have

$$\sum_n \to \int \frac{d^3q}{(2\pi)^3}$$

and the completeness condition becomes

$$\int \frac{d^3q}{(2\pi)^3}\left[u^*_\mathbf{q}(\mathbf{r})u_\mathbf{q}(\mathbf{r}') + v^*_\mathbf{q}(\mathbf{r}')v_\mathbf{q}(\mathbf{r})\right] = \delta(\mathbf{r}-\mathbf{r}')$$

The BCS model: Hamiltonian

$$\mathcal{H} = \mathcal{H}_{kin} + \mathcal{H}_{int}$$

$$\mathcal{H}_{kin} = \sum_{\alpha} \int d^3 r \Psi^\dagger(\mathbf{r},\alpha) \hat{H}_e \Psi(\mathbf{r},\alpha) \qquad \mathcal{H}_{int} = -\frac{V}{2} \sum_{\alpha,\beta} \int d^3 r \Psi^\dagger(\mathbf{r},\alpha)\Psi^\dagger(\mathbf{r},\beta)\Psi(\mathbf{r},\beta)\Psi(\mathbf{r},\alpha)$$

Define an effective mean-field Hamiltonian:

$$\begin{aligned} \mathcal{H}_{eff} \;\; &= \;\; \int d^3 r \sum_{\alpha} \left[\Psi^\dagger(\mathbf{r},\alpha) \hat{H}_e \Psi(\mathbf{r},\alpha) + U(\mathbf{r}) \Psi^\dagger(\mathbf{r},\alpha)\Psi(\mathbf{r},\alpha) \right] \\ &+ \int d^3 r \left[\Delta(\mathbf{r}) \Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow) + \Delta^*(\mathbf{r}) \Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow) + H_0(\mathbf{r}) \right] \end{aligned}$$

How to find right quasiparticles and effective fields $U(\boldsymbol{r})$ and $\Delta(\boldsymbol{r})$?

(1) This Hamiltonian is diagonal in the new operators $\gamma_{n,\alpha}$ and $\gamma^\dagger_{n,\alpha}$

$$\mathcal{H}_{eff} = E_g + \sum_{n,\alpha} \epsilon_n \gamma^\dagger_{n,\alpha} \gamma_{n,\alpha} \ ,$$

(2) The statistically averaged energy $\langle \mathcal{H}_{eff} \rangle$ has a minimum at the same wave functions as the average energy of the true Hamiltonian $\langle \mathcal{H} \rangle$. Moreover, these minimal values coincide.

$$\langle \mathcal{H}_{eff} \rangle = \mathcal{Z}^{-1} \sum_k \left\langle \psi_k^\dagger | \mathcal{H}_{eff} | \psi_k \right\rangle \exp(-E_k/T)$$

$$\mathcal{H}_{eff} \psi_k = E_k \psi_k$$

$$\mathcal{Z} = \sum_k \exp(-E_k/T)$$

The BCS model: Bogolubov - de Gennes equations

To find right wave functions of quasiparticles let us try to fulfil condition (1):

$$[\mathcal{H}_{eff}, \Psi(\mathbf{r},\uparrow)]_- = -\left[\hat{H}_e + U(\mathbf{r})\right]\Psi(\mathbf{r},\uparrow) - \Delta(\mathbf{r})\Psi^\dagger(\mathbf{r},\downarrow)$$

$$[\mathcal{H}_{eff}, \Psi(\mathbf{r},\downarrow)]_- = -\left[\hat{H}_e + U(\mathbf{r})\right]\Psi(\mathbf{r},\downarrow) + \Delta(\mathbf{r})\Psi^\dagger(\mathbf{r},\uparrow)$$

$$[\mathcal{H}_{eff}, \gamma_{n,\alpha}]_- = -\epsilon_n \gamma_{n,\alpha}$$

$$[\mathcal{H}_{eff}, \gamma^\dagger_{n,\alpha}]_- = \epsilon_n \gamma^\dagger_{n,\alpha}$$

Substitute Bogolubov transformation here

And compare

We obtain the Bogolubov - de Gennes equations:

$$\left[\hat{H}_e + U(\mathbf{r})\right]u(\mathbf{r}) + \Delta(\mathbf{r})v(\mathbf{r}) = \epsilon u(\mathbf{r})$$

$$-\left[\hat{H}^*_e + U(\mathbf{r})\right]v(\mathbf{r}) + \Delta^*(\mathbf{r})u(\mathbf{r}) = \epsilon v(\mathbf{r})$$

$$\check{\Omega}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \epsilon_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

$$\check{\Omega} = \begin{pmatrix} \hat{H}_e + U & \Delta \\ \Delta^* & -[\hat{H}^*_e + U] \end{pmatrix}$$

Orthogonality condition:

$$\int \left[u^*_m(\mathbf{r})u_n(\mathbf{r}) + v^*_m(\mathbf{r})v_n(\mathbf{r})\right] d^3r = \delta_{mn}$$

$$\int \left[u^*_{\mathbf{q}_1}(\mathbf{r})u_{\mathbf{q}_2}(\mathbf{r}) + v^*_{\mathbf{q}_1}(\mathbf{r})v_{\mathbf{q}_2}(\mathbf{r})\right] d^3r = (2\pi)^3\delta(\mathbf{q}_1 - \mathbf{q}_2)$$

The BCS model: Bogolubov - de Gennes equations

Important properties of the Bogolubov – de Gennes equations:

(1) If the vector

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad \text{belongs to an energy } \epsilon_n$$

then the vector

$$\begin{pmatrix} v_n^* \\ -u_n^* \end{pmatrix} \quad \text{belongs to the energy } -\epsilon_n$$

(2) The Bogolubov – de Gennes equations conserves the quasiparticle flow:

$$\operatorname{div} \mathbf{P} = 0$$

$$\mathbf{P} = u^* \left(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}\right) u + u \left(i\hbar\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}\right) u^*$$
$$-v^* \left(-i\hbar\boldsymbol{\nabla} + \frac{e}{c}\mathbf{A}\right) v - v \left(i\hbar\boldsymbol{\nabla} + \frac{e}{c}\mathbf{A}\right) v^*$$

The BCS model: the self-consistency equation

To fulfil condition (2) we write

$$\begin{aligned}\langle\mathcal{H}\rangle &= \sum_{\alpha}\int d^3r\left\langle\Psi^\dagger(\mathbf{r},\alpha)\hat{H}_e\Psi(\mathbf{r},\alpha)\right\rangle\\ &\quad -\frac{V}{2}\sum_{\alpha,\beta}\int d^3r\,\langle\Psi^\dagger(\mathbf{r},\alpha)\Psi^\dagger(\mathbf{r},\beta)\Psi(\mathbf{r},\beta)\Psi(\mathbf{r},\alpha)\rangle\end{aligned}$$

According to the Wick's theorem:

$$\left\langle\Psi^\dagger(\mathbf{r}_1,\alpha)\Psi^\dagger(\mathbf{r}_2,\beta)\Psi(\mathbf{r}_3,\gamma)\Psi(\mathbf{r}_4,\delta)\right\rangle =$$

$$\begin{aligned}&\left\langle\Psi^\dagger(\mathbf{r}_1,\alpha)\Psi^\dagger(\mathbf{r}_2,\beta)\right\rangle\left\langle\Psi(\mathbf{r}_3,\gamma)\Psi(\mathbf{r}_4,\delta)\right\rangle\\ &+\left\langle\Psi^\dagger(\mathbf{r}_1,\alpha)\Psi(\mathbf{r}_4,\delta)\right\rangle\left\langle\Psi^\dagger(\mathbf{r}_2,\beta)\Psi(\mathbf{r}_3,\gamma)\right\rangle\\ &-\left\langle\Psi^\dagger(\mathbf{r}_1,\alpha)\Psi(\mathbf{r}_3,\gamma)\right\rangle\left\langle\Psi^\dagger(\mathbf{r}_2,\beta)\Psi(\mathbf{r}_4,\delta)\right\rangle\end{aligned}$$

Abrikosov, A.A., Gorkov, L.P. and Dzyaloshinski, I.E. (1964)
Method of Quantum Field Theory in Statistical Physics.
Prentice-Hall, New Jersey.

$$\langle\Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow)\rangle\neq 0 \text{ and } \langle\Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\rangle\neq 0$$

due to the Cooper pairing, while

$$\langle\Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\uparrow)\rangle = 0 \text{ and } \langle\Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\downarrow)\rangle = 0$$

because pairing occurs for particles with opposite spins.
Assume also that

$$\langle\Psi^\dagger(\mathbf{r},\uparrow)\Psi(\mathbf{r},\downarrow)\rangle = 0 \text{ and } \langle\Psi^\dagger(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\rangle = 0$$

due to the absence of magnetic interaction.

Then the Wick's theorem gives:

$$\sum_{\alpha,\beta}\langle\Psi^\dagger(\mathbf{r},\alpha)\Psi^\dagger(\mathbf{r},\beta)\Psi(\mathbf{r},\beta)\Psi(\mathbf{r},\alpha)\rangle =$$

$$\begin{aligned}&\langle\Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow)\rangle\langle\Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\rangle\\ &+\langle\Psi^\dagger(\mathbf{r},\downarrow)\Psi^\dagger(\mathbf{r},\uparrow)\rangle\langle\Psi(\mathbf{r},\uparrow)\Psi(\mathbf{r},\downarrow)\rangle\\ &+\sum_{\alpha,\beta}\langle\Psi^\dagger(\mathbf{r},\alpha)\Psi(\mathbf{r},\alpha)\rangle\langle\Psi^\dagger(\mathbf{r},\beta)\Psi(\mathbf{r},\beta)\rangle\\ &-\sum_{\alpha,\beta}\langle\Psi^\dagger(\mathbf{r},\alpha)\Psi(\mathbf{r},\beta)\rangle\langle\Psi^\dagger(\mathbf{r},\beta)\Psi(\mathbf{r},\alpha)\rangle\end{aligned}$$

The BCS model: the self-consistency equation

Using the anticommutation relations we obtain:

$$\sum_{\alpha,\beta} \left\langle \Psi^\dagger(\mathbf{r},\alpha)\Psi^\dagger(\mathbf{r},\beta)\Psi(\mathbf{r},\beta)\Psi(\mathbf{r},\alpha)\right\rangle = 2\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow)\right\rangle \left\langle \Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\right\rangle + 2\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi(\mathbf{r},\uparrow)\right\rangle \left\langle \Psi^\dagger(\mathbf{r},\downarrow)\Psi(\mathbf{r},\downarrow)\right\rangle$$

The variation of the true energy:

$$\begin{aligned}\delta\langle\mathcal{H}\rangle = & \sum_\alpha \int d^3r\,\delta\left\langle \Psi^\dagger(\mathbf{r},\alpha)\hat{H}_e\Psi(\mathbf{r},\alpha)\right\rangle \\ & -V\int d^3r\,\left[\left(\delta\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow)\right\rangle\right)\left\langle \Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\right\rangle\right. \\ & \left.+\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow)\right\rangle\left(\delta\left\langle \Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\right\rangle\right)\right] \\ & -V\int d^3r\,\left[\left(\delta\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi(\mathbf{r},\uparrow)\right\rangle\right)\left\langle \Psi^\dagger(\mathbf{r},\downarrow)\Psi(\mathbf{r},\downarrow)\right\rangle\right. \\ & \left.+\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi(\mathbf{r},\uparrow)\right\rangle\left(\delta\left\langle \Psi^\dagger(\mathbf{r},\downarrow)\Psi(\mathbf{r},\downarrow)\right\rangle\right)\right]\end{aligned}$$

The variation of the effective energy:

$$\begin{aligned}\delta\langle\mathcal{H}_{eff}\rangle = & \sum_\alpha \int d^3r\,\delta\left\langle \Psi^\dagger(\mathbf{r},\alpha)\left[\hat{H}_e + U(\mathbf{r})\right]\Psi(\mathbf{r},\alpha)\right\rangle \\ & + \int d^3r\,\left[\Delta(\mathbf{r})\delta\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow)\right\rangle + \Delta^*(\mathbf{r})\delta\left\langle \Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\right\rangle\right]\end{aligned}$$

Compare and obtain the self-consistency equations:

$$\boxed{\begin{aligned}U(\mathbf{r}) &= -V\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi(\mathbf{r},\uparrow)\right\rangle = -V\left\langle \Psi^\dagger(\mathbf{r},\downarrow)\Psi(\mathbf{r},\downarrow)\right\rangle \\ \Delta(\mathbf{r}) &= -V\left\langle \Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\right\rangle = V\left\langle \Psi(\mathbf{r},\uparrow)\Psi(\mathbf{r},\downarrow)\right\rangle \\ \Delta^*(\mathbf{r}) &= -V\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow)\right\rangle = V\left\langle \Psi^\dagger(\mathbf{r},\downarrow)\Psi^\dagger(\mathbf{r},\uparrow)\right\rangle\end{aligned}}$$

The BCS model: the self-consistency equation

How to calculate averages in the self-consistency equations?

$$U(\mathbf{r}) = -V\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi(\mathbf{r},\uparrow)\right\rangle = -V\left\langle \Psi^\dagger(\mathbf{r},\downarrow)\Psi(\mathbf{r},\downarrow)\right\rangle$$

$$\Delta(\mathbf{r}) = -V\left\langle \Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\right\rangle = V\left\langle \Psi(\mathbf{r},\uparrow)\Psi(\mathbf{r},\downarrow)\right\rangle$$

$$\Delta^*(\mathbf{r}) = -V\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow)\right\rangle = V\left\langle \Psi^\dagger(\mathbf{r},\downarrow)\Psi^\dagger(\mathbf{r},\uparrow)\right\rangle$$

To do this we use:

$$\begin{aligned}\left\langle \gamma^\dagger_{n,\alpha}\gamma_{m,\beta}\right\rangle &= \delta_{nm}\delta_{\alpha\beta}f_n\\ \left\langle \gamma_{n,\alpha}\gamma_{m,\beta}\right\rangle &= \left\langle \gamma^\dagger_{n,\alpha}\gamma^\dagger_{m,\beta}\right\rangle = 0\end{aligned}$$

where f_n is the distribution function.

In equilibrium, it is the Fermi function

$$f_n = \frac{1}{e^{\epsilon_n/T}+1}$$

The resulting self-consistency equations take the form:

$$\boxed{\begin{aligned}\Delta(\mathbf{r}) &= V\sum_n (1 - f_{n\uparrow} - f_{n\downarrow})\, u_n(\mathbf{r})v_n^*(\mathbf{r}) = V\sum_n (1-2f_n)\, u_n(\mathbf{r})v_n^*(\mathbf{r})\\ \Delta^*(\mathbf{r}) &= V\sum_n (1 - f_{n\uparrow} - f_{n\downarrow})\, u_n^*(\mathbf{r})v_n(\mathbf{r}) = V\sum_n (1-2f_n)\, u_n^*(\mathbf{r})v_n(\mathbf{r})\\ U(\mathbf{r}) &= -V\sum_n \left[|u_n(\mathbf{r})|^2 f_n + |v_n(\mathbf{r})|^2(1-f_n)\right]\end{aligned}}$$

The BCS model: electron density and final form of the Hamiltonian

The electron density takes the form:

$$n = 2\left\langle \Psi^\dagger(\mathbf{r},\uparrow)\Psi(\mathbf{r},\uparrow)\right\rangle = 2\sum_n \left[|u_n(\mathbf{r})|^2 f_n + |v_n(\mathbf{r})|^2(1-f_n)\right]$$

It is a combination of

a particle contribution

$$2\sum_n |u_n(\mathbf{r})|^2 f_n$$

a hole contribution

$$2\sum_n |v_n(\mathbf{r})|^2(1-f_n)$$

The average energy of the state with effective field $\Delta(\boldsymbol{r})$

$$\langle \mathcal{H}\rangle_\Delta = \int d^3r \left[\sum_\alpha \left\langle \Psi^\dagger(\mathbf{r},\alpha)\hat{H}_e\Psi(\mathbf{r},\alpha)\right\rangle_\Delta - \frac{|\Delta(\mathbf{r})|^2}{V}\right]$$

The final form of the effective Hamiltonian:

$$\begin{aligned}\mathcal{H}_{eff} \quad &= \int d^3r \left[\sum_\alpha \Psi^\dagger(\mathbf{r},\alpha)\hat{H}_e\Psi(\mathbf{r},\alpha) + \frac{|\Delta(\mathbf{r})|^2}{V}\right] \\ &+ \int d^3r \left[\Delta(\mathbf{r})\Psi^\dagger(\mathbf{r},\uparrow)\Psi^\dagger(\mathbf{r},\downarrow) + \Delta^*(\mathbf{r})\Psi(\mathbf{r},\downarrow)\Psi(\mathbf{r},\uparrow)\right]\end{aligned}$$

Including the effective field $U(\boldsymbol{r})$ into the chemical potential we obtain the final form of BDG equations:

$$-\frac{\hbar^2}{2m}\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2 u - E_F u + \Delta v = \epsilon u$$

$$\frac{\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar c}\mathbf{A}\right)^2 v + E_F v + \Delta^* u = \epsilon v$$

The BCS model: Observables. Energy spectrum and coherence factors.

Consider a homogeneous superconductor with no magnetic field: $\Delta = |\Delta| e^{i\chi}$

$$\begin{aligned}\left[-\frac{\hbar^2}{2m}\nabla^2 - \mu\right] u(\mathbf{r}) + \Delta v(\mathbf{r}) &= \epsilon u(\mathbf{r}) \\ -\left[-\frac{\hbar^2}{2m}\nabla^2 - \mu\right] v(\mathbf{r}) + \Delta^* u(\mathbf{r}) &= \epsilon v(\mathbf{r})\end{aligned}$$

$$\mu = \hbar^2 k_F^2 / 2m$$

We look for a solution in the form: $u = e^{\frac{i}{2}\chi} U_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}\ ,\ v = e^{-\frac{i}{2}\chi} V_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}$

$$\begin{aligned}\xi_{\mathbf{q}} U_{\mathbf{q}} + |\Delta| V_{\mathbf{q}} &= \epsilon_{\mathbf{q}} U_{\mathbf{q}} \\ -\xi_{\mathbf{q}} V_{\mathbf{q}} + |\Delta| U_{\mathbf{q}} &= \epsilon_{\mathbf{q}} V_{\mathbf{q}}\end{aligned}$$

with $\xi_{\mathbf{q}} = \frac{\hbar^2}{2m}\left[q^2 - k_F^2\right]$

The system has a nontrivial solution if

$$\boxed{\epsilon_{\mathbf{q}} = \pm\sqrt{\xi_{\mathbf{q}}^2 + |\Delta|^2}}$$

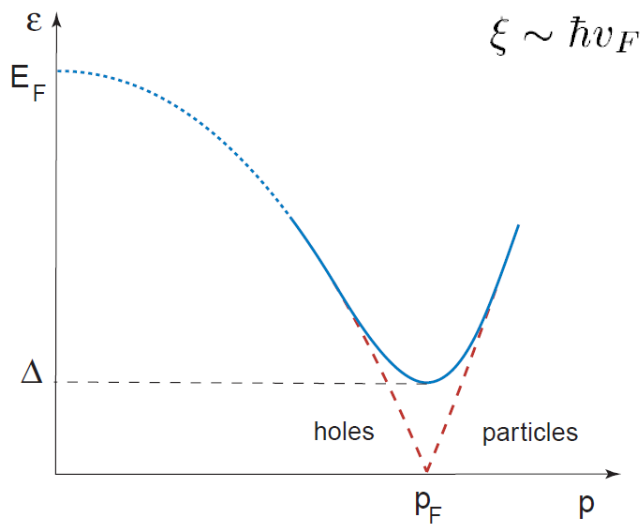
$\epsilon > 0$ gives the eigen energies of quasiparticle states

$$U_{\mathbf{q}} = \frac{1}{\sqrt{2}}\left(1 + \frac{\xi_{\mathbf{q}}}{\epsilon_{\mathbf{q}}}\right)^{1/2},\ V_{\mathbf{q}} = \frac{1}{\sqrt{2}}\left(1 - \frac{\xi_{\mathbf{q}}}{\epsilon_{\mathbf{q}}}\right)^{1/2}$$

- **the coherence factors**

the Cooper pair size:

$$\xi \sim \hbar v_F / |\Delta|$$



Quasiparticle group velocity: $\frac{d\epsilon}{d(\hbar q)} = v_g$

$$q^2 = k_F^2 \pm (2m/\hbar^2)\sqrt{\epsilon^2 - |\Delta|^2}$$

$$v_g = \frac{\hbar q}{m}\frac{\xi_{\mathbf{q}}}{\sqrt{\xi_{\mathbf{q}}^2 + |\Delta|^2}} = v_F \frac{\xi_{\mathbf{q}}}{\sqrt{\xi_{\mathbf{q}}^2 + |\Delta|^2}} = \pm v_F \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\epsilon}$$

The BCS model: Observables. Density of states (DOS).

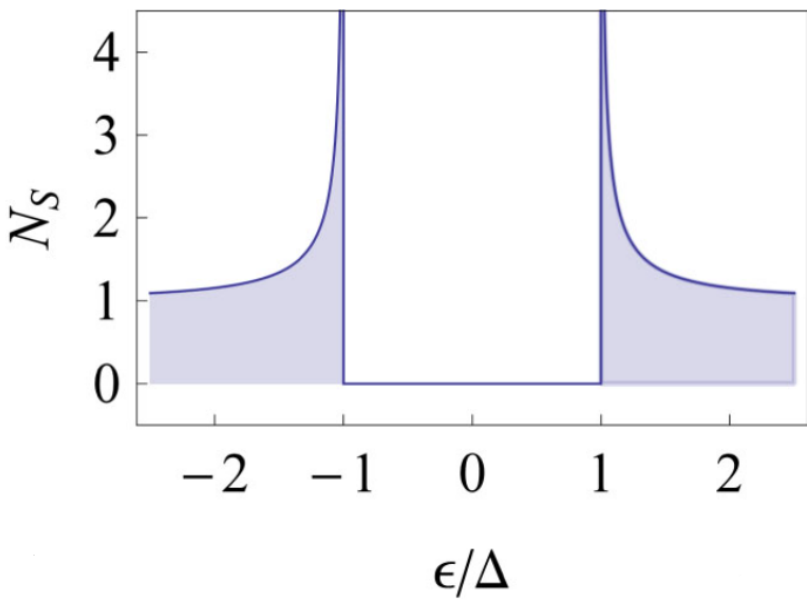
$$\boxed{N(\epsilon) = \frac{dn_\alpha(q)}{d\epsilon}}$$

$n_\alpha(q)$ - is the number of states per unit volume and per spin for particles with momenta up to $\hbar q$

For $\epsilon > |\Delta|$

$$\begin{aligned} N(\epsilon) &= \frac{1}{2}\left|\frac{d}{d\epsilon}\left(\frac{q^3}{3\pi^2}\right)\right| = \frac{q^2}{2\pi^2}\left|\frac{d\epsilon}{dq}\right|^{-1} \\ &= \frac{mq}{2\pi^2\hbar^2}\frac{\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} \approx N(0)\frac{\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} \end{aligned}$$

$$N(0) = \frac{mk_F}{2\pi^2\hbar^2}$$



The BCS model: Observables. Structure of quasiparticle wave functions.

for a given energy ϵ

$$q^2 = k_F^2 + \frac{2m}{\hbar^2}\xi_{\mathbf{q}} = k_F^2 \pm \frac{2m}{\hbar^2}\sqrt{\epsilon^2 - |\Delta|^2}$$

$$q = \pm q_{\pm} \qquad q_{\pm} = k_F \pm \frac{1}{\hbar v_F}\sqrt{\epsilon^2 - |\Delta|^2}$$

The BDG equation in a homogeneous superconductor has a general solution:

$$\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} e^{\frac{i}{2}\chi}U_+^+ \\ e^{-\frac{i}{2}\chi}V_+^+ \end{pmatrix} e^{iq_+x} + B\begin{pmatrix} e^{\frac{i}{2}\chi}U_+^- \\ e^{-\frac{i}{2}\chi}V_+^- \end{pmatrix} e^{iq_-x} + C\begin{pmatrix} e^{\frac{i}{2}\chi}U_-^+ \\ e^{-\frac{i}{2}\chi}V_-^+ \end{pmatrix} e^{-iq_+x} + D\begin{pmatrix} e^{\frac{i}{2}\chi}U_-^- \\ e^{-\frac{i}{2}\chi}V_-^- \end{pmatrix} e^{-iq_-x}$$

$$U_\pm^+ = \frac{1}{\sqrt{2}}\left(1 + \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\epsilon}\right)^{1/2}, \quad V_\pm^+ = \frac{1}{\sqrt{2}}\left(1 - \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\epsilon}\right)^{1/2} \qquad U_\pm^- = \frac{1}{\sqrt{2}}\left(1 - \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\epsilon}\right)^{1/2}, \quad V_\pm^- = \frac{1}{\sqrt{2}}\left(1 + \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\epsilon}\right)^{1/2}$$

In the normal state $\Delta = 0$ the wave function for a given energy ϵ becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{iq_+x} + B\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{iq_-x} + C\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-iq_+x} + D\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-iq_-x}$$

It consists of

a particle with the group velocity along the momentum

$$u = Ae^{iq_+x} + Ce^{-iq_+x}$$

a hole with the group velocity opposite to the momentum

$$v = Be^{iq_-x} + De^{-iq_-x}$$

The BCS model: Observables. The energy gap.

$$\Delta(\mathbf{r}) = V \sum (1 - 2f_n)\, u_n(\mathbf{r}) v_n^*(\mathbf{r})$$

$$uv^* = e^{i\chi} U_{\mathbf{q}} V_{\mathbf{q}} = \frac{\Delta}{2\epsilon_{\mathbf{q}}}$$

$$\Delta = V \sum_{\mathbf{q}} (1 - 2f_{\mathbf{q}}) \frac{\Delta}{2\epsilon_{\mathbf{q}}}$$

We replace the sum with the integral

$$\sum_{\mathbf{q}} = \int_{q=0}^{q=\infty} \frac{d^3q}{(2\pi)^3} = \int_{\xi=-E_F}^{\xi=+\infty} \frac{mq}{2\pi^2\hbar^2}\, d\xi_{\mathbf{q}} \approx N(0) \int_{-\infty}^{+\infty} d\xi_{\mathbf{q}}$$

and notice that, in equilibrium,

$$1 - 2f_{\mathbf{q}} = \tanh\left(\frac{\epsilon_{\mathbf{q}}}{2T}\right)$$

Moreover,

$$\epsilon_{\mathbf{q}}\, d\epsilon_{\mathbf{q}} = \xi_{\mathbf{q}}\, d\xi_{\mathbf{q}}$$

When ξ varies from $-\infty$ to $+\infty$, the energy varies from Δ to $+\infty$ taking each value twice. Therefore, the self-consistency equation takes the form

$$\Delta = V \sum_{\mathbf{q}} (1 - 2f_{\mathbf{q}}) \frac{\Delta}{2\epsilon_{\mathbf{q}}} = N(0)V \int_{|\Delta|}^{\infty} \frac{\Delta}{\sqrt{\epsilon^2 - |\Delta|^2}} \tanh\left(\frac{\epsilon}{2T}\right) d\epsilon$$

Assume that

$$V_\epsilon = \begin{cases} V, & \epsilon < E_c \\ 0, & \epsilon > E_c \end{cases}$$

Then we obtain **the gap equation**

$$1 = \lambda \int_{|\Delta|}^{E_c} \frac{1}{\sqrt{\epsilon^2 - |\Delta|^2}} \tanh\left(\frac{\epsilon}{2T}\right) d\epsilon$$

The BCS model: Observables. The critical temperature.

$$1 = \lambda \int_{|\Delta|}^{E_c} \frac{1}{\sqrt{\epsilon^2 - |\Delta|^2}} \tanh\left(\frac{\epsilon}{2T}\right) d\epsilon$$

Equation for the critical temperature T_c at which Δ vanishes:

$$1 = \lambda \int_0^{E_c} \tanh\left(\frac{\epsilon}{2T_c}\right) \frac{d\epsilon}{\epsilon}$$

This reduces to

$$\frac{1}{\lambda} = \int_0^{E_c/2T_c} \frac{\tanh x}{x} dx$$

The integral

$$\int_0^a \frac{\tanh x}{x} dx = \ln(aB)$$

Here $B = 4\gamma/\pi \approx 2.26$ where $\gamma = e^C \approx 1.78$ and $C = 0.577\ldots$ is the Euler constant. Therefore,

$$\boxed{T_c = (2\gamma/\pi)E_c e^{-1/\lambda} \approx 1.13 E_c e^{-1/\lambda}}$$

At zero temperature:

$$\frac{1}{\lambda} = \int_{|\Delta|}^{E_c} \frac{d\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} = \mathrm{Arcosh}\left(\frac{E_c}{|\Delta|}\right) \approx \ln(2E_c/|\Delta|)$$

$$\boxed{|\Delta| \equiv \Delta(0) = (\pi/\gamma)T_c \approx 1.76 T_c}$$

The BCS model: Observables. The condensation energy.

Average energy per volume in the superconducting state:

$$\langle \mathcal{H} \rangle_\Delta = \int d^3r \left[\sum_\alpha \left\langle \Psi^\dagger(\mathbf{r},\alpha) \hat{H}_e \Psi(\mathbf{r},\alpha) \right\rangle_\Delta - \frac{|\Delta(\mathbf{r})|^2}{V} \right]$$

$$\int d^3r \sum_\alpha \left\langle \Psi^\dagger(\mathbf{r},\alpha) \hat{H}_e \Psi(\mathbf{r},\alpha) \right\rangle = \sum_{n,m,\alpha} \left[\langle \gamma^\dagger_{m\alpha} \gamma_{n\alpha} \rangle \left(\int u^*_m \hat{H}_e u_n \, d^3r \right) + \langle \gamma_{m\alpha} \gamma^\dagger_{n\alpha} \rangle \left(\int v_m \hat{H}_e v^*_n \, d^3r \right) \right]$$

$$= 2 \sum_n \left[f_n \left(\int u^*_n \hat{H}_e u_n \, d^3r \right) + (1 - f_n) \left(\int v_n \hat{H}_e v^*_n \, d^3r \right) \right] = 2 \sum_n \left[\epsilon_n \left(f_n |u_n|^2 - (1 - f_n)|v_n|^2 \right) + \Delta v_n u^*_n (1 - 2f_n) \right]$$

$$= 2 \sum_n \left[\epsilon_n f_n - \epsilon_n |v_n|^2 \right] + \frac{2|\Delta|^2}{V}$$

Result:

$$\boxed{\langle \mathcal{H} \rangle_\Delta = 2 \sum_n \epsilon_n f_n - 2 \sum_n \epsilon_n |v_n|^2 + \frac{|\Delta|^2}{V}}$$

energy of excitations

Energy of the ground state
measured from the normal state

The BCS model: Observables. The condensation energy.

Energy of the ground state – **the condensation energy**:

$$\begin{aligned}\mathcal{E}_\Delta &= -2\sum_n \epsilon_n |v_n|^2 + \frac{|\Delta|^2}{V} = -\sum_n (\epsilon_n - \xi_n) + \frac{|\Delta|^2}{V} \\ &= -2N(0)\int_0^{E_c}\left(\sqrt{\xi^2+|\Delta|^2}-\xi\right)d\xi + \frac{|\Delta|^2}{V}\end{aligned}$$

$$\begin{aligned}\int_0^{E_c}\left(\sqrt{\xi^2+|\Delta|^2}-\xi\right)d\xi &= \frac{1}{2}\left[\xi\sqrt{\xi^2+|\Delta|^2}-\xi^2+|\Delta|^2\ln(\xi+\sqrt{\xi^2+|\Delta|^2})\right]_0^{E_c} \\ &= \frac{|\Delta|^2}{4}+\frac{|\Delta|^2}{2}\ln\left(\frac{2E_c}{|\Delta|}\right) = \frac{|\Delta|^2}{4}+\frac{|\Delta|^2}{2N(0)V}\end{aligned}$$

Here we use that $|\Delta \ll E_c$

Result:

$$\boxed{\mathcal{E}_\Delta = -\frac{N(0)|\Delta(0)|^2}{2}}$$

The BCS model: Observables. The current.

The quantum mechanical expression for the current density:

$$\mathbf{j} = \frac{e}{2m}\sum_{\alpha}\left\{\Psi^{\dagger}(\mathbf{r},\alpha)\left[\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)\Psi(\mathbf{r},\alpha)\right] + \left[\left(i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)\Psi^{\dagger}(\mathbf{r},\alpha)\right]\Psi(\mathbf{r},\alpha)\right\}$$

$$\mathbf{j} = \frac{e}{m}\sum_{n}\left[f_n\, u_n^*(\mathbf{r})\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)u_n(\mathbf{r}) + (1-f_n)\, v_n(\mathbf{r})\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)v_n^*(\mathbf{r}) + c.c.\right]$$

In the presence of the supercurrent pairs have nonzero total momentum: $\Delta = |\Delta| e^{i\mathbf{k}\cdot\mathbf{r}}$

$$u(\mathbf{r}) = e^{i(\mathbf{q}+\mathbf{k}/2)\cdot\mathbf{r}}U_{\mathbf{q}}\,,\;\; v(\mathbf{r}) = e^{i(\mathbf{q}-\mathbf{k}/2)\cdot\mathbf{r}}V_{\mathbf{q}}$$

The energy spectrum takes the form:

$$\epsilon_{\mathbf{q}} = \hbar\mathbf{q}\cdot\mathbf{v}_s + \sqrt{\xi_{\mathbf{q}}^2 + |\Delta|^2} = \hbar\mathbf{q}\cdot\mathbf{v}_s + \epsilon_{\mathbf{q}}^{(0)}$$

$\mathbf{v}_s = \dfrac{\hbar\mathbf{k}}{2m}$ is the superconducting velocity.

$$U_{\mathbf{q}} = \frac{1}{\sqrt{2}}\left(1 + \frac{\xi_{\mathbf{q}}}{\epsilon_{\mathbf{q}}^{(0)}}\right)^{1/2},\; V_{\mathbf{q}} = \frac{1}{\sqrt{2}}\left(1 - \frac{\xi_{\mathbf{q}}}{\epsilon_{\mathbf{q}}^{(0)}}\right)^{1/2}$$

the gap vanishes for excitations with $\mathbf{q}$ antiparallel to $\mathbf{v}_s$ if $v_s \geq v_c$

$\boxed{v_c = |\Delta|/p_F}$ **is the critical velocity**

The BCS model: Observables. The current.

$$\mathbf{j} \;=\; \frac{2\hbar e}{m}\sum_{\mathbf{q}}\left[\left(\mathbf{q}+\frac{\mathbf{k}}{2}\right)f_{\mathbf{q}}U_{\mathbf{q}}^2-\left(\mathbf{q}-\frac{\mathbf{k}}{2}\right)(1-f_{\mathbf{q}})V_{\mathbf{q}}^2\right] \;=\; \frac{\hbar e}{m}\mathbf{k}\sum_{\mathbf{q}}\left[f_{\mathbf{q}}U_{\mathbf{q}}^2+(1-f_{\mathbf{q}})V_{\mathbf{q}}^2\right]+\frac{2\hbar e}{m}\sum_{\mathbf{q}}\mathbf{q}\left[f_{\mathbf{q}}U_{\mathbf{q}}^2-(1-f_{\mathbf{q}})V_{\mathbf{q}}^2\right]$$

$$f_{\mathbf{q}} \equiv f(\epsilon_{\mathbf{q}}) = \frac{1}{e^{\epsilon_{\mathbf{q}}/T}+1}$$

$$\mathbf{j} \;=\; \frac{\hbar e}{m}\mathbf{k}\sum_{\mathbf{q}}\left[f(\epsilon_{\mathbf{q}}^{(0)})U_{\mathbf{q}}^2+(1-f(\epsilon_{\mathbf{q}}^{(0)}))V_{\mathbf{q}}^2\right] \longleftarrow \quad \mathbf{j}_0 = \frac{\hbar e n}{2m}\mathbf{k} = ne\mathbf{v}_s \text{ flow of } \textit{all} \text{ particles with the velocity } \mathbf{v}_s.$$

$$+\frac{2\hbar e}{m}\sum_{\mathbf{q}}\mathbf{q}\left[f(\epsilon_{\mathbf{q}}^{(0)})U_{\mathbf{q}}^2-(1-f(\epsilon_{\mathbf{q}}^{(0)}))V_{\mathbf{q}}^2\right] \longleftarrow \quad \text{vanishes after summation over directions of } \mathbf{q}.$$

$$+\frac{\hbar e}{m}\mathbf{k}\sum_{\mathbf{q}}\left[f(\epsilon_{\mathbf{q}})-f(\epsilon_{\mathbf{q}}^{(0)})\right]\left[U_{\mathbf{q}}^2-V_{\mathbf{q}}^2\right] \longleftarrow \quad \text{vanishes after summation over directions of } \mathbf{q}.$$

$$+\frac{2\hbar e}{m}\sum_{\mathbf{q}}\mathbf{q}\left[f(\epsilon_{\mathbf{q}})-f(\epsilon_{\mathbf{q}}^{(0)})\right]\left[U_{\mathbf{q}}^2+V_{\mathbf{q}}^2\right] \longleftarrow \quad -\mathbf{j}_{norm} = \frac{2\hbar e}{m}\sum_{\mathbf{q}}\mathbf{q}\left[f(\epsilon_{\mathbf{q}}^{(0)})-f(\epsilon_{\mathbf{q}})\right]$$

$$\mathbf{j}_{norm} = en_{norm}\mathbf{v}_s$$

The total current is:

$$\mathbf{j} = e(n-n_{norm})\mathbf{v}_s = en_s\mathbf{v}_s$$

$n_s = n - n_{norm}$ is the density of superconducting electrons

The BCS model: Observables. The current.

The calculation of the normal current for small superconducting velocity:

$$\mathbf{j}_{norm} = \frac{2\hbar e}{m} \sum_{\mathbf{q}} \mathbf{q} \left[f(\epsilon_{\mathbf{q}}^{(0)}) - f(\epsilon_{\mathbf{q}}) \right]$$

$$f(\epsilon_{\mathbf{q}}^{(0)}) - f(\epsilon_{\mathbf{q}}) = \frac{1}{e^{\sqrt{\xi_{\mathbf{q}}^2 + |\Delta|^2}/T} + 1} - \frac{1}{e^{(\sqrt{\xi_{\mathbf{q}}^2 + |\Delta|^2} + \hbar \mathbf{q} \mathbf{v}_s)/T} + 1}$$

$$\mathbf{j}_{norm} = -\frac{2\hbar^2 e}{m} \sum_{\mathbf{q}} \mathbf{q} (\mathbf{q} \cdot \mathbf{v}_s) \frac{df(\epsilon_{\mathbf{q}}^{(0)})}{d\epsilon_{\mathbf{q}}^{(0)}} = -\frac{2\hbar^2 e}{3m} \mathbf{v}_s \sum_{\mathbf{q}} \mathbf{q}^2 \frac{df(\epsilon_{\mathbf{q}}^{(0)})}{d\epsilon_{\mathbf{q}}^{(0)}}$$

$$n_{norm} = -\frac{2p_F^2}{3m} \sum_{\mathbf{q}} \frac{df(\epsilon_{\mathbf{q}}^{(0)})}{d\epsilon_{\mathbf{q}}^{(0)}} = -\frac{2p_F^2}{3m} N(0) \int_{-\infty}^{\infty} \frac{df(\epsilon_{\mathbf{q}}^{(0)})}{d\epsilon_{\mathbf{q}}^{(0)}} d\xi_{\mathbf{q}} = -n \int_{-\infty}^{\infty} \frac{df(\epsilon_{\mathbf{q}}^{(0)})}{d\epsilon_{\mathbf{q}}^{(0)}} d\xi_{\mathbf{q}} = -2n \int_{|\Delta|}^{\infty} \frac{\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} \frac{df(\epsilon)}{d\epsilon} d\epsilon$$

For $\Delta = 0$ we have $n_{norm} = n$.

The BCS model: Negative energies.

$$\begin{aligned}
\Delta &= V\sum_n (1-2f_n)\,u_n(\epsilon_n)v_n^*(\epsilon_n) \\
&= \frac{V}{2}\sum_{n,\epsilon>0} \left[[1-2f(\epsilon_n)]\,u_n(\epsilon_n)v_n^*(\epsilon_n) - [1-2f(-\epsilon_n)]\,v_n^*(-\epsilon_n)u_n(-\epsilon_n)\right] \\
&= \frac{V}{2}\left[\sum_{n,\epsilon>0}[1-2f(\epsilon_n)]\,\frac{\Delta}{2\epsilon_n} - \sum_{n,\epsilon>0}[1-2f(-\epsilon_n)]\,\frac{\Delta}{2\epsilon_n}\right] \\
&= \frac{V}{2}\sum_{n,\text{all }\epsilon}[1-2f(\epsilon_n)]\,\frac{\Delta}{2\epsilon_n}
\end{aligned}$$

Let us introduce the even and odd combinations

$$f_1 = -f(\epsilon)+f(-\epsilon)\,,\; f_2 = -f(\epsilon)+[1-f(-\epsilon)] = 1-f(\epsilon)-f(-\epsilon)$$

$$\Delta = \frac{V}{2}\sum_{n,\text{all }\epsilon} f_1(\epsilon_n)\frac{\Delta}{2\epsilon_n}$$

In equilibrium $f_1 = 1-2f(\epsilon) = \tanh\frac{\epsilon}{2T}$ $\qquad f_2 = 0$

$$\boxed{\Delta = V\sum_{n,\epsilon>0}\frac{\Delta}{2\epsilon}\tanh\frac{\epsilon}{2T}}$$

Problems to Section 1

1.1 Calculate the ground state energy of the normal Fermi gas. Express it via E_F.

1.2 Derive the completeness conditions $\sum_n [u_n^*(\mathbf{r})u_n(\mathbf{r}') + v_n^*(\mathbf{r}')v_n(\mathbf{r})] = \delta(\mathbf{r}-\mathbf{r}')$ and $\sum_n [u_n^*(\mathbf{r})v_n(\mathbf{r}') - u_n^*(\mathbf{r}')v_n(\mathbf{r})] = 0$ from the Fermi commutation rules.

1.3 Derive the orthogonality condition.

1.4 Derive the expression for the quasiparticle flow $\boldsymbol{P}$ from the BDG equations and show that $\boldsymbol{P}$ is conserved.

1.5 Find the energy spectrum and the coherence factors for the order parameter $\Delta = |\Delta|e^{i\mathbf{k}\cdot\mathbf{r}}$

1.6 Derive the gap equation which determines the dependence of $|\Delta|$ on v_s for the order parameter in the form $\Delta = |\Delta|e^{i\mathbf{k}\cdot\mathbf{r}}$

1.7 Find the temperature dependence of the gap at $T \to T_c$.