Microscopic theory of mesoscopic superconductivity

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International summer school at Beijing Institute of Technology, 16th – 26th July 2023

Outline of the 3d section – NIS interface

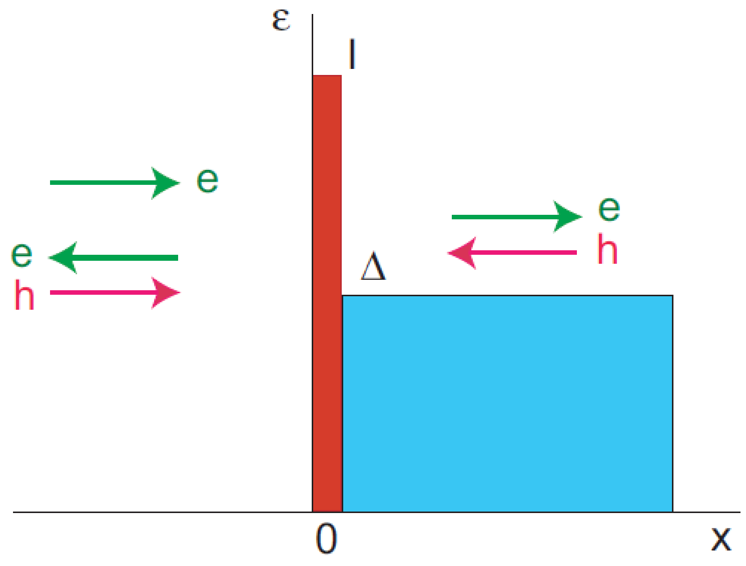
- Transmission through the barrier

- Current through the NIS junction

  Normal tunnel resistance
  Landauer formula
  Tunnel current
  Excess current
  NS Andreev current. Current conversion

- SIS junction

NIS interface. Transmission through the barrier.



$$e^{ik_y y + ik_z z}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E_x\right] u + \Delta v = \epsilon u$$

$$-\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E_x\right] v + \Delta^* u = \epsilon v$$

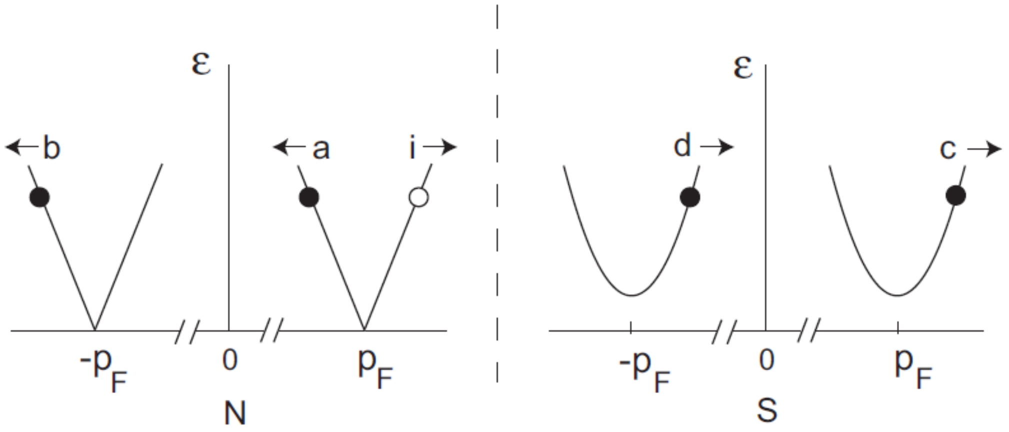
$$E_x = E_F - \frac{\hbar^2(k_y^2 + k_z^2)}{2m}$$

A particle is incident from the normal region on the left. Its wave function in the normal part:

$$e^{iq_+(N)x}\begin{pmatrix} U_+^+(N) \\ V_+^+(N) \end{pmatrix} + ae^{iq_-(N)x}\begin{pmatrix} U_+^-(N) \\ V_+^-(N) \end{pmatrix} + be^{-iq_+(N)x}\begin{pmatrix} U_-^+(N) \\ V_-^+(N) \end{pmatrix}$$

$$q_\pm(N) = k_x \pm \frac{\epsilon}{\hbar v_x}$$

NIS interface. Transmission through the barrier.



In the N region:

$$e^{iq_+(N)x}\begin{pmatrix} U_+^+(N) \\ V_+^+(N) \end{pmatrix} + a e^{iq_-(N)x}\begin{pmatrix} U_+^-(N) \\ V_+^-(N) \end{pmatrix} + b e^{-iq_+(N)x}\begin{pmatrix} U_-^+(N) \\ V_-^+(N) \end{pmatrix}$$

$$U_\pm^+(N) = 1,\ V_\pm^+(N) = 0,\ U_\pm^-(N) = 0,\ V_\pm^-(N) = 1$$

In the S region:

$$c e^{iq_+(S)x}\begin{pmatrix} U_+^+(S) \\ V_+^+(S) \end{pmatrix} + d e^{-iq_-(S)x}\begin{pmatrix} U_-^-(S) \\ V_-^-(S) \end{pmatrix} \qquad q_\pm(S) = k_x \pm \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\hbar v_x}$$

$$U_\pm^+(S) = V_\pm^-(S) \equiv U,\ U_\pm^-(S) = V_\pm^+(S) \equiv V \qquad U^2 - V^2 = \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\epsilon},\ UV = \frac{|\Delta|}{2\epsilon}$$

NIS interface. Transmission through the barrier.

The BdG equation conserves the quasiparticle flow: $\mathrm{div}\,\mathbf{P}=0$

$$\mathbf{P}=u^*\left(-i\hbar\boldsymbol{\nabla}-\frac{e}{c}\mathbf{A}\right)u+u\left(i\hbar\boldsymbol{\nabla}-\frac{e}{c}\mathbf{A}\right)u^*$$
$$-v^*\left(-i\hbar\boldsymbol{\nabla}+\frac{e}{c}\mathbf{A}\right)v-v\left(i\hbar\boldsymbol{\nabla}+\frac{e}{c}\mathbf{A}\right)v^*\ .$$

The quasiparticle flow on the left $P_L=2q_+(N)\left(1-|b|^2\right)-2q_-(N)|a|^2$

The quasiparticle flow on the right

$$\begin{aligned}P_R &= q_+(S)|c|^2\left(U^2-V^2\right)+q_-(S)|d|^2\left(U^2-V^2\right)\\ &\quad +c^*d\left[q_-(S)\left(VU-UV\right)+q_+(S)\left(VU-UV\right)\right]e^{-iq_-(S)x-iq_+(S)x}+c.c\\ &= 2\left(q_+(S)|c|^2+q_-(S)|d|^2\right)\left(U^2-V^2\right)\end{aligned}$$

The conservation of the flow yields:

$$q_+(N)\left(1-|b|^2\right)-q_-(N)|a|^2=\left[q_+(S)|c|^2+q_-(S)|d|^2\right]\frac{\sqrt{\epsilon^2-|\Delta|^2}}{\epsilon}$$

$$q_+(N)|b|^2+q_-(N)|a|^2+\left[q_+(S)|c|^2+q_-(S)|d|^2\right]\frac{\sqrt{\epsilon^2-|\Delta|^2}}{\epsilon}=q_+(N)$$

Note: $\hbar q_\pm(S)\dfrac{\sqrt{\epsilon^2-|\Delta|^2}}{\epsilon}=mv_{g,\pm}(S)$ $\qquad \hbar q_\pm(N)=mv_{g,\pm}(N)$

The flow implies the conservation of quasiparticle current probabilities:

$$\boxed{v_{g+}(N)|b|^2+v_{g-}(N)|a|^2+v_{g+}(S)|c|^2+v_{g-}(S)|d|^2=v_{g+}(N)}$$

NIS interface. Transmission and reflection probabilities.

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E_x + I\delta(x)\right] u + \Delta v = \epsilon u$$

$$-\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E_x + I\delta(x)\right] v + \Delta^* u = \epsilon v$$

The boundary conditions at the barrier $x = 0$ are

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}_L = \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}_R \qquad \frac{d}{dx}\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}_R - \frac{d}{dx}\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}_L = 2|k_x|Z\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$$

where the barrier strength is $\boxed{Z = \dfrac{mI}{\hbar^2|k_x|}}$

Consider first the case $\epsilon > |\Delta|$.

There are four independent solutions. (1) The wave function that has incident particle on the left of the barrier, (2) The wave function with incident particle on the right, (3) Incident hole on the left, and (4) Incident hole on the right.

NIS interface. Transmission and reflection probabilities.

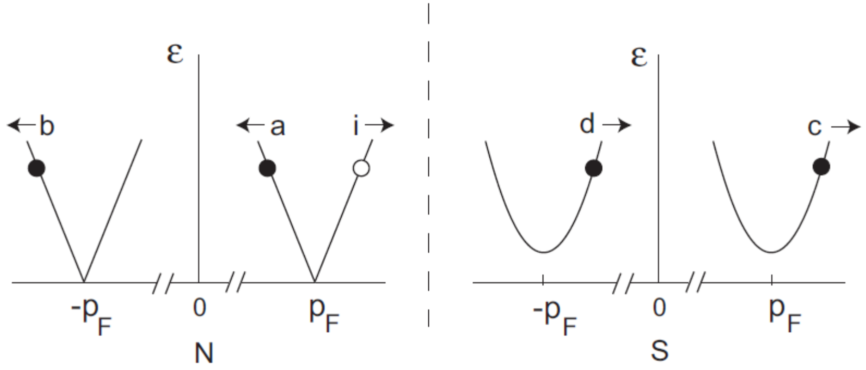
(1):

on the left:

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_L = e^{iq_+(N)x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + ae^{iq_-(N)x}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + be^{-iq_+(N)x}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

on the right:

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_R = ce^{iq_+(S)x}\begin{pmatrix} U \\ V \end{pmatrix} + de^{-iq_-(S)x}\begin{pmatrix} V \\ U \end{pmatrix}$$



the boundary conditions yield:

$$1 + b = cU + dV \qquad q_-(S)Vd - q_+(S)Uc - q_+(N)b + q_+(N) = 2i|k_x|Z(1+b)$$

$$a = cV + dU \qquad q_-(S)Ud - q_+(S)Vc + q_-(N)a = 2i|k_x|Za$$

NIS interface. Transmission and reflection probabilities.

We notice that $q_+ + q_- \approx 2k_x$ while $q_+ - q_- \sim \Delta/\hbar v_F$ so that $q_+ - q_- \ll q_+ + q_-$ and also $q_+ - q_- \ll k_F Z$. As a result, in the semiclassical approximation,

$$\begin{aligned} a &= \frac{UV}{U^2 + (U^2 - V^2)Z^2} \\ b &= -\frac{(U^2 - V^2)(Z^2 + iZ)}{U^2 + (U^2 - V^2)Z^2} \\ c &= \frac{(1 - iZ)U}{U^2 + (U^2 - V^2)Z^2} \\ d &= \frac{iZV}{U^2 + (U^2 - V^2)Z^2} \end{aligned}$$

Limiting cases:

NIN interface

$V = 0$, $U = 1$, and $a = d = 0$

$$b = -\frac{iZ}{1 + iZ}, \quad c = \frac{1}{1 + iZ}$$

$$|b|^2 = \frac{Z^2}{1 + Z^2}, \quad |c|^2 = \frac{1}{1 + Z^2}$$

No barrier

$Z = 0$ and $b = d = 0$

$$a = \frac{V}{U}, \quad c = \frac{1}{U}$$

NIS interface. Transmission and reflection probabilities.

(2):

on the right:

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_R = e^{-iq_+(S)x}\begin{pmatrix} U \\ V \end{pmatrix} + a_2 e^{-iq_-(S)x}\begin{pmatrix} V \\ U \end{pmatrix} + b_2 e^{iq_+(S)x}\begin{pmatrix} U \\ V \end{pmatrix}$$

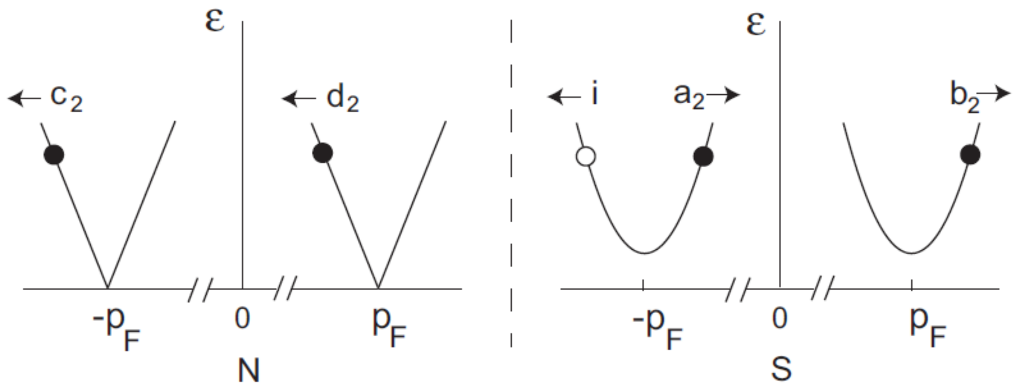
on the left:

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_L = c_2 e^{-iq_+(N)x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + d_2 e^{iq_-(N)x}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the boundary conditions yield:

$$U + a_2 V + b_2 U = c_2$$
$$V + a_2 U + b_2 V = d_2$$

$$q_+(N)c_2 - q_+(S)U - q_-(S)V a_2 + q_+(S)U b_2 = 2i|k_x|Z c_2$$
$$-q_-(N)d_2 - q_+(S)V - q_-(S)U a_2 + q_+(S)V b_2 = 2i|k_x|Z d_2$$



NIS interface. Transmission and reflection probabilities.

As a result, we find $a_2 = -a^* \; , \; b_2 = b^*$

$$c_2 = \frac{v_g(S)}{v_g(N)} c^*$$

$$d_2 = -\frac{v_g(S)}{v_g(N)} d^*$$

The conservation of flow in the direct process (1) complies with the flow in the reverse process (2):

$$v_g(S)|a_2|^2 + v_g(S)|b_2|^2 + v_g(N)|c_2|^2 + v_g(N)|d_2|^2 = v_g(S)$$

Since the group velocities and the DOS are coupled through $\frac{N_S}{N_N} = \frac{v_F}{v_g(S)}$

we find

$$(v_F|c_2|^2)N_S = (v_g|c|^2)N_N \; , \; (v_F|d_2|^2)N_S = (v_g|d|^2)N_N$$

$$(v_g(S)|a_2|^2)N_S = (v_F|a|^2)N_N \; , \; (v_g(S)|b_2|^2)N_S = (v_F|b|^2)N_N$$

which express the detailed balance of particle flow within an energy interval dE.

NIS interface. Transmission and reflection probabilities.

The state (3) with an incident hole in the normal region has

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_L = e^{-iq_-(N)x}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_3 e^{-iq_+(N)x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b_3 e^{iq_-(N)x}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It contains normally reflected hole with the amplitude b_3, and Andreev reflected particle with the amplitude a_3. On the right of the barrier it has transmitted hole c_3 and a particle d_3

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_R = c_3 e^{-iq_-(S)x}\begin{pmatrix} V \\ U \end{pmatrix} + d_3 e^{iq_+(S)x}\begin{pmatrix} U \\ V \end{pmatrix}$$

$$\begin{aligned} a &= \frac{UV}{U^2 + (U^2 - V^2)Z^2} \\ b &= -\frac{(U^2 - V^2)(Z^2 + iZ)}{U^2 + (U^2 - V^2)Z^2} \\ c &= \frac{(1 - iZ)U}{U^2 + (U^2 - V^2)Z^2} \\ d &= \frac{iZV}{U^2 + (U^2 - V^2)Z^2} \end{aligned}$$

NIS interface. Transmission and reflection probabilities.

The state (4) has an incident hole on the right

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_R = e^{iq_-(S)x} \begin{pmatrix} V \\ U \end{pmatrix} + a_4 e^{iq_+(S)x} \begin{pmatrix} U \\ V \end{pmatrix} + b_4 e^{-iq_-(S)x} \begin{pmatrix} V \\ U \end{pmatrix}$$

On the left of the barrier (in the N region)

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_L = c_4 e^{iq_-(N)x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + d_4 e^{-iq_+(N)x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$a_2 = -a^* \,,\ b_2 = b^*$$

$$\begin{aligned} c_2 &= \frac{v_g(S)}{v_g(N)} c^* \\ d_2 &= -\frac{v_g(S)}{v_g(N)} d^* \end{aligned}$$

NIS interface. Transmission and reflection probabilities.

Consider now the case $\epsilon < |\Delta|$. For the state (1) with $k_x > 0$ we have on the right of the barrier in the superconducting region only the decaying waves

$$\left(\begin{array}{c} u(x) \\ v(x) \end{array}\right)_R = c e^{i\tilde{q}_+(S)x} \left(\begin{array}{c} \tilde{U} \\ \tilde{V} \end{array}\right) + d e^{-i\tilde{q}_-(S)x} \left(\begin{array}{c} \tilde{V} \\ \tilde{U} \end{array}\right)$$

$$\tilde{q}_\pm(S) = k_x \pm \frac{im}{\hbar^2 k_x}\sqrt{|\Delta|^2 - \epsilon^2} \qquad \tilde{U} = \frac{1}{\sqrt{2}}\left(1 + i\frac{\sqrt{|\Delta|^2 - \epsilon^2}}{\epsilon}\right)^{1/2}, \ \tilde{V} = \frac{1}{\sqrt{2}}\left(1 - i\frac{\sqrt{|\Delta|^2 - \epsilon^2}}{\epsilon}\right)^{1/2}$$

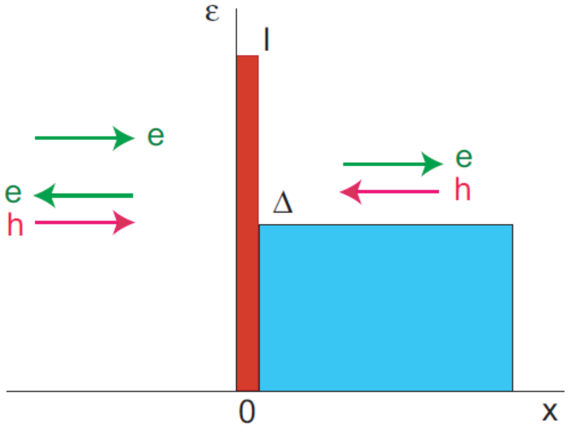
The coefficients *a,b,c,d* are obtained by the replacement *U* and *V* with $\tilde{U}$ and $\tilde{V}$ in the previous expressions:

$$\begin{aligned} a &= \frac{UV}{U^2 + (U^2 - V^2)Z^2} \\ b &= -\frac{(U^2 - V^2)(Z^2 + iZ)}{U^2 + (U^2 - V^2)Z^2} \\ c &= \frac{(1 - iZ)U}{U^2 + (U^2 - V^2)Z^2} \\ d &= \frac{iZV}{U^2 + (U^2 - V^2)Z^2} \end{aligned}$$

$$|a|^2 = \frac{|\Delta|^2}{\epsilon^2 + (|\Delta|^2 - \epsilon^2)(1 + 2Z^2)^2}$$

$$|a|^2 + |b|^2 = 1$$

Current through the NIS junction.

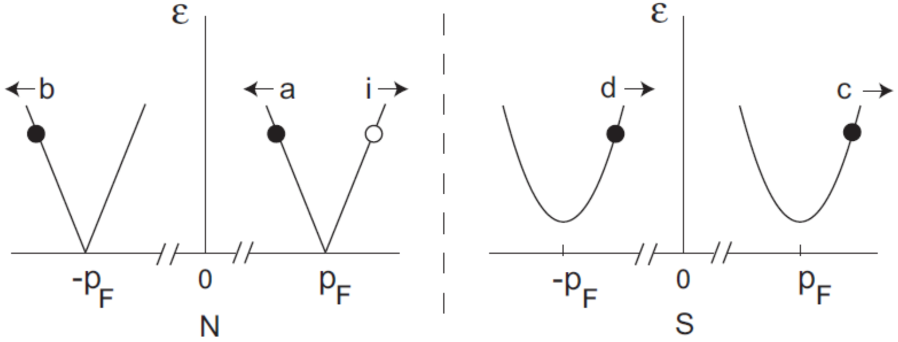


Apply a voltage V across the interface, the current is:

$$\mathbf{j} = -\frac{i\hbar e}{m} \sum_{n=1,\ldots,4;\epsilon,\mathbf{p}} \left[f_{n,\epsilon,\mathbf{p}} \left(u^*_{n,\epsilon,\mathbf{p}} \nabla u_{n,\epsilon,\mathbf{p}} \right) + \left(1 - f_{n,\epsilon,\mathbf{p}}\right) \left(v_{n,\epsilon,\mathbf{p}} \nabla v^*_{n,\epsilon,\mathbf{p}} \right) - c.c \right]$$

It is convenient to calculate the current at the normal state of the interface

State (1):



The state (1) has particles coming from the normal region. For this reason, their distribution function corresponds to that in the normal region $f = f_\epsilon(N)$.

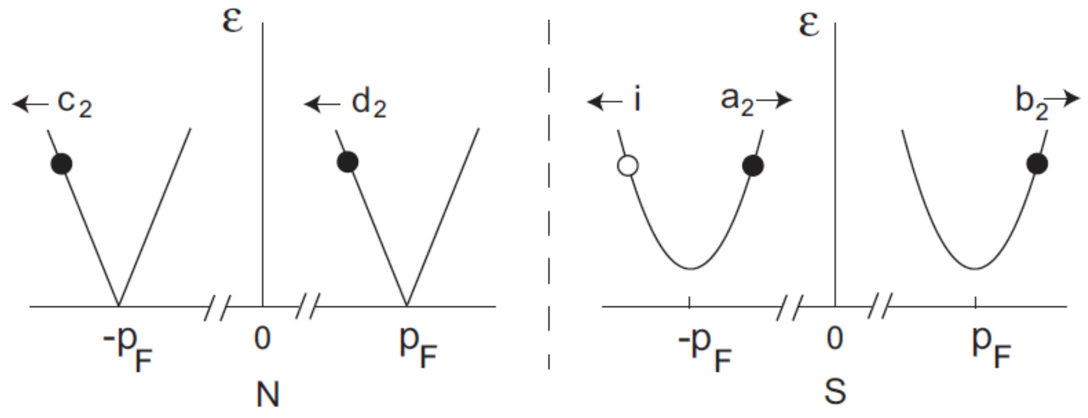
The sum over all possible states (1) is calculated as $\sum_{n=1} = \int_{k_x>0} N_N \, d\epsilon \, \frac{d\Omega_{\mathbf{k}}}{4\pi}$

$$-iu^*_{1,\epsilon,\mathbf{p}} \nabla_x u_{1,\epsilon,\mathbf{p}} + c.c. = 2k_x \left(1 - |b|^2\right) \;, \; -iv_{1,\epsilon,\mathbf{p}} \nabla_x v^*_{1,\epsilon,\mathbf{p}} + c.c. = -2k_x |a|^2$$

$$-iu^*_{1,\epsilon,-\mathbf{p}} \nabla_x u_{1,\epsilon,-\mathbf{p}} + c.c. = -2k_x \left(1 - |b|^2\right) \;, \; -iv_{1,\epsilon,-\mathbf{p}} \nabla_x v^*_{1,\epsilon,-\mathbf{p}} + c.c. = 2k_x |a|^2$$

Current through the NIS junction.

State (2):



The state (2) has particles coming from the superconducting region. For this reason, their distribution function corresponds to that in the superconductor $f = f_\epsilon(S)$.

The sum over all possible states (2) is calculated as $\sum_{n=2} = \int_{k_x>0} N_S \, d\epsilon \, \frac{d\Omega_{\mathbf{k}}}{4\pi}$

$$-iu^*_{2,\epsilon,\mathbf{p}} \nabla_x u_{2,\epsilon,\mathbf{p}} + c.c. = -2k_x |c_2|^2 \, , \; -iv_{2,\epsilon,\mathbf{p}} \nabla_x v^*_{2,\epsilon,\mathbf{p}} + c.c. = -2k_x |d_2|^2$$

$$-iu^*_{2,\epsilon,-\mathbf{p}} \nabla_x u_{2,\epsilon,-\mathbf{p}} + c.c. = 2k_x |c_2|^2 \, , \; -iv_{2,\epsilon,-\mathbf{p}} \nabla_x v^*_{2,\epsilon,-\mathbf{p}} + c.c. = 2k_x |d_2|^2$$

Current through the NIS junction.

The states (3) and (4) can be obtained from (1) and (2), respectively, by changing signs of ϵ and k_x in $q_\pm$ and in U, V with simultaneous change of 1 to 0 and vice versa (which also corresponds to the change of U into V and vice versa) in the normal region coherence factors.

State (3): $$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_L = e^{-iq_-(N)x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_3 e^{-iq_+(N)x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b_3 e^{iq_-(N)x} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$q_\pm(N) = k_x \pm \frac{\epsilon}{\hbar v_x}$$

State (1): $$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix}_L = e^{iq_+(N)x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a e^{iq_-(N)x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b e^{-iq_+(N)x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{j} &= -\frac{i\hbar e}{m} \sum_{n=1,2;\epsilon} \left[f_{n,\epsilon} \left(u^*_{n,\epsilon,\mathbf{p}} \nabla u_{n,\epsilon,\mathbf{p}} \right) + (1 - f_{n,\epsilon}) \left(v_{n,\epsilon,\mathbf{p}} \nabla v^*_{n,\epsilon,\mathbf{p}} \right) - c.c \right] \\
&\quad -\frac{i\hbar e}{m} \sum_{n=3,4;\epsilon} \left[f_{n,\epsilon} \left(u^*_{n,\epsilon,\mathbf{p}} \nabla u_{n,\epsilon,\mathbf{p}} \right) + (1 - f_{n,\epsilon}) \left(v_{n,\epsilon,\mathbf{p}} \nabla v^*_{n,\epsilon,\mathbf{p}} \right) - c.c \right] \\
&= -\frac{i\hbar e}{m} \sum_{n=1,2;\epsilon>0} \left[f_{n,\epsilon} \left(u^*_{n,\epsilon,\mathbf{p}} \nabla u_{n,\epsilon,\mathbf{p}} \right) + (1 - f_{n,\epsilon}) \left(v_{n,\epsilon,\mathbf{p}} \nabla v^*_{n,\epsilon,\mathbf{p}} \right) - c.c \right] \\
&\quad -\frac{i\hbar e}{m} \sum_{n=1,2;\epsilon>0} \left[f_{n,-\epsilon} \left(v_{n,-\epsilon,-\mathbf{p}} \nabla v^*_{n,-\epsilon,-\mathbf{p}} \right) + (1 - f_{n,-\epsilon}) \left(u^*_{n,-\epsilon,-\mathbf{p}} \nabla u_{n,-\epsilon,-\mathbf{p}} \right) - c.c \right]
\end{aligned}$$

Current through the NIS junction.

$$
\begin{aligned}
I_{NIS} &= \frac{2\hbar e S}{m}\int_{\epsilon>0} d\epsilon \int \frac{d\Omega_{\mathbf{k}}}{4\pi}\, k_x \left[\left(f_\epsilon(N)[1-|b|^2]-[1-f_\epsilon(N)]|a|^2\right) N_N\right.\\
&\quad -\left(f_\epsilon(S)|c_2|^2+[1-f_\epsilon(S)]|d_2|^2\right) N_S\\
&\quad +\left(f_{-\epsilon}(N)|a|^2-[1-|b|^2][1-f_{-\epsilon}(N)]\right) N_N\\
&\quad \left.+\left(f_{-\epsilon}(S)|d_2|^2+[1-f_{-\epsilon}(S)]|c_2|^2\right) N_S\right]\\
&= \frac{2\hbar e S}{m}\int_{\epsilon>0} d\epsilon \int_{k_x>0} \frac{d\Omega_{\mathbf{k}}}{4\pi}\, k_x \left(\left[1-|b|^2+|a|^2\right]\left[f_\epsilon(N)+f_{-\epsilon}(N)-1\right] N_N\right.\\
&\quad \left.-\left[|c_2|^2-|d_2|^2\right]\left[f_\epsilon(S)+f_{-\epsilon}(S)-1\right] N_S\right)
\end{aligned}
$$

In the normal region, $f_\epsilon(N) = f_0(\epsilon - eV)$ where $f_0(\epsilon) = \dfrac{1}{e^{\epsilon/T}+1}$ $\Rightarrow$

$f_\epsilon(N) + f_{-\epsilon}(N) - 1 = f_0(\epsilon - eV) + f_0(-\epsilon - eV) - 1 = f_0(\epsilon - eV) - f_0(\epsilon + eV)$

In the superconducting region $f_\epsilon(S) = f_0(\epsilon)$. $\Rightarrow$ $f_\epsilon(S) + f_{-\epsilon}(S) - 1 = 0$

Current through the NIS junction.

The current becomes

$$\begin{aligned} I_{NIS} &= \frac{2\hbar e S}{m} \int_{\epsilon>0} d\epsilon \int_{k_x>0} \frac{d\Omega_{\mathbf{k}}}{4\pi} k_x \left[1 - |b|^2 + |a|^2\right] \left[f_\epsilon(N) + f_{-\epsilon}(N) - 1\right] N_N \\ &= \frac{\hbar e S}{m} \int_{-\infty}^{\infty} d\epsilon \int_{k_x>0} \frac{d\Omega_{\mathbf{k}}}{4\pi} k_x \left[1 - |b|^2 + |a|^2\right] \left[f_\epsilon(N) + f_{-\epsilon}(N) - 1\right] N_N \ . \end{aligned}$$

We note that

$$\int_{-\infty}^{\infty} B(\epsilon) \left[f_0(\epsilon - eV) - f_0(\epsilon + eV)\right] d\epsilon = 2 \int_{-\infty}^{\infty} B(\epsilon) \left[f_0(\epsilon - eV) - f_0(\epsilon)\right] d\epsilon$$

where $B(\epsilon) = 1 - |b|^2 + |a|^2$ is an even function. Finally, if Z is independent of the incident angle,

$$I_{NIS} = AeN(0)v_F S \int_{-\infty}^{\infty} d\epsilon \left[1 - |b|^2 + |a|^2\right] \left[f_0(\epsilon - eV) - f_0(\epsilon)\right]$$

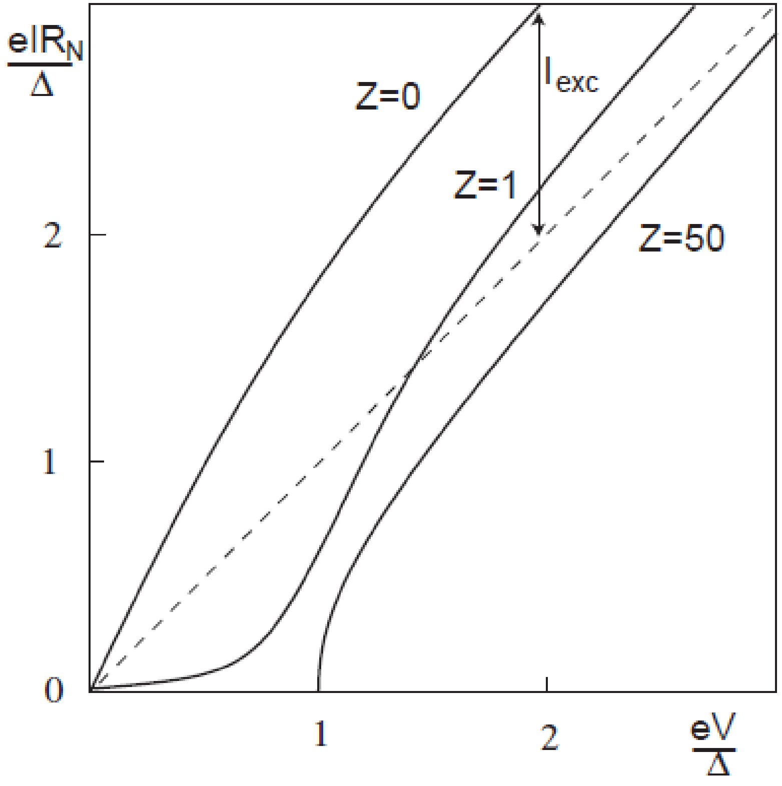
where $A \sim 1$ is a constant that depends on the geometry.

The factor $1 - |b|^2 + |a|^2$ plays the role of the transmission coefficient for particle-hole reflection at the NIS interface. It is by the Andreev reflection coefficient $|a|^2$ *larger* that that in the normal state.

Current through the NIS junction.

The current-voltage characteristics of SIN junction at low temperatures.

The picture is taken from the work by G.E. Blonder, M. Tinkham and T.M. Klapwijk, Phys.Rev. B **25** 4515 (1982)



Current through the NIS junction. Normal tunnel resistance.

At first, assume that the superconductor is in the normal state $|\Delta| = 0.$

Then $|a|^2 = 0$ and $|b|^2 = \frac{Z^2}{1+Z^2}$

$$I_{NIN} = \frac{AeN(0)v_F S}{1+Z^2}\int_{-\infty}^{\infty}[f_0(\epsilon - eV) - f_0(\epsilon)]\, d\epsilon = -\frac{Ae^2N(0)v_F SV}{1+Z^2}\int_{-\infty}^{\infty}\frac{df_0}{d\epsilon}\,d\epsilon = \frac{V}{R_N}$$

$$\frac{1}{R_N} = \frac{Ae^2N(0)v_F S}{1+Z^2}$$

Then the current through the NIS junction can be written as:

$$I_{NIS} = \frac{1+Z^2}{eR_N}\int_{-\infty}^{\infty} d\epsilon\ \left[1 - |b|^2 + |a|^2\right]\left[f_0(\epsilon - eV) - f_0(\epsilon)\right]$$

For $Z = 0$ we have $\frac{1}{R_N} = Ae^2N(0)v_F S \equiv \frac{1}{R_{\mathrm{Sh}}}$ - Sharvin resistance

Current through the NIS junction. Landauer formula.

Consider a point contact between two normal metals in more detail. We assume that the barrier is absent so that the electrons fly freely (ballistically) through the constriction from one electrode to another. The current through the constriction is

$$I = e \sum_{p_x>0;p_y,p_z} [|v_x| f_0(\epsilon - eV) - |v_x| f_0(\epsilon)] = 2e \sum_{p_y,p_z} \int_0^\infty \frac{dp_x}{2\pi\hbar} \frac{\partial \epsilon_{p_y,p_z}(p_x)}{\partial p_x} [f_0(\epsilon - eV) - f_0(\epsilon)]$$

$$\boxed{I = \frac{2e}{h} \sum_{p_y,p_z} \int_{-E_F}^{\infty} d\epsilon \, [f_0(\epsilon - eV) - f_0(\epsilon)] = \frac{2e^2}{h} V \sum_{p_y,p_z} \int_{-\infty}^{\infty} \frac{df_0(\epsilon)}{d\epsilon} d\epsilon = \frac{2e^2 N_>}{h} V}$$

- Landauer formula

$N_>$ is the number of states with $p_x > 0$ that go through the constriction.

The Landauer formula shows that the conductance $G = \left(\frac{2e^2}{h}\right) N$ of the point constriction is an integer number of the quantum conductance $G_0 = 2e^2/h$

$$R_0 = \frac{1}{G_0} = \frac{h}{2e^2} \approx 12.9 \; k\Omega$$

- quantum of resistance

For a constriction with a barrier:

$$\boxed{G_{NIN} = \frac{2e^2}{h} \sum_{n=1}^{N_>} \mathcal{T}_n}$$

- Landauer-Büttiker formula

$$\mathcal{T}_n = 1 - |b_n|^2 = \frac{1}{1 + Z_n^2}$$

- the transmission coefficient for the mode *n*

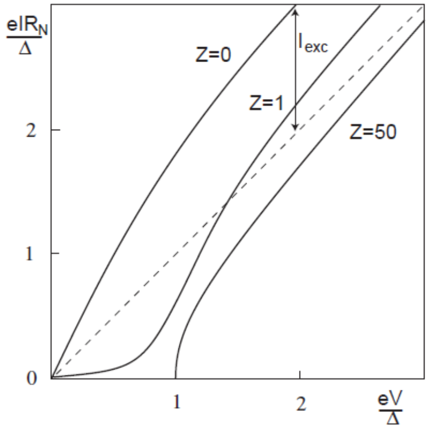
Current through the NIS junction. Tunnel current.

Consider a junction with a strong barrier $Z^2 \gg 1$. For $|\epsilon| > |\Delta|$ we find

$$|b|^2 = 1 - \frac{1}{Z^2(U^2 - V^2)} = 1 - \frac{\epsilon}{Z^2\sqrt{\epsilon^2 - |\Delta|^2}} = 1 - \frac{N_S(\epsilon)}{Z^2 N(0)}$$

$$|a|^2 \sim Z^{-4}$$

$$\begin{aligned} a &= \frac{UV}{U^2 + (U^2 - V^2)Z^2} \\ b &= -\frac{(U^2 - V^2)(Z^2 + iZ)}{U^2 + (U^2 - V^2)Z^2} \\ c &= \frac{(1 - iZ)U}{U^2 + (U^2 - V^2)Z^2} \\ d &= \frac{iZV}{U^2 + (U^2 - V^2)Z^2} \end{aligned}$$



For $|\epsilon| < |\Delta|$ $\quad |a|^2 \sim Z^{-4} \quad |b|^2 = 1$

$$|a|^2 = \frac{|\Delta|^2}{\epsilon^2 + (|\Delta|^2 - \epsilon^2)(1 + 2Z^2)^2}$$

$$|a|^2 + |b|^2 = 1$$

Therefore, we obtain for the current:

$$\boxed{I_{NIS} = \frac{1}{eR_N}\int_{-\infty}^{\infty} \frac{N_S(\epsilon)}{N(0)}\left[f_0(\epsilon - eV) - f_0(\epsilon)\right] d\epsilon}$$

where we put $\dfrac{N_S(\epsilon)}{N(0)} = \dfrac{\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}}\Theta\left(\epsilon^2 - |\Delta|^2\right)$

For low temperatures $T \ll T_c$

$$\boxed{I_{NIS} = \frac{\sqrt{(eV)^2 - |\Delta|^2}}{eR_N}\Theta(eV - |\Delta|)}$$

Current through the NIS junction. Excess current.

For large voltages, $eV \gg |\Delta|$ the integral in $\quad I_{NIS} = \dfrac{1+Z^2}{eR_N}\displaystyle\int_{-\infty}^{\infty} d\epsilon\ \left[1-|b|^2+|a|^2\right]\left[f_0(\epsilon-eV)-f_0(\epsilon)\right]$

is determined by energies of the order of eV. Indeed, for $\epsilon \gg |\Delta|$

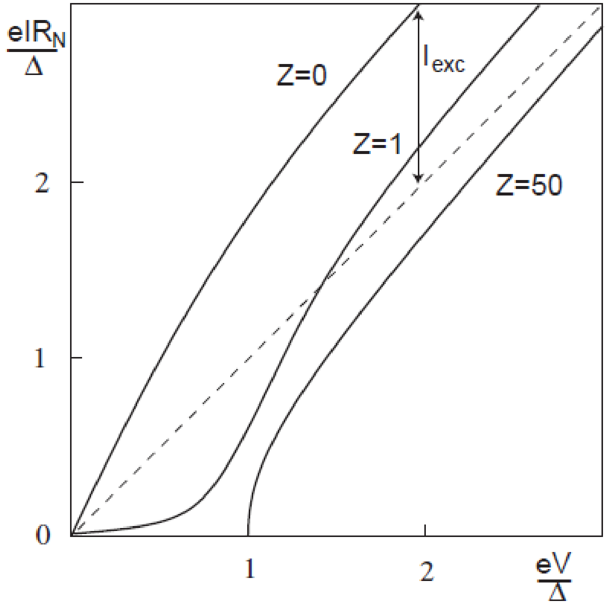
$$|a|^2 = \frac{|\Delta|^2}{\epsilon^2(1+Z^2)^2}\,,\quad |b|^2 = \frac{Z^2}{1+Z^2} - \frac{|\Delta|^2 Z^2}{4\epsilon^2(1+Z^2)^2}$$

Therefore $I \approx V/R_N$. The curve $I(V)$ for large V goes parallel to the Ohm's law, but it is shifted by a constant current which is called the excess current

$$\begin{aligned} I_{exc}(V) &= I_{NIS}(V) - I_N(V) \\ &= \frac{1}{eR_N}\int_{-\infty}^{\infty} d\epsilon\ \left(\left[1+Z^2\right]\left[1-|b|^2+|a|^2\right]-1\right)\left[f_0(\epsilon-eV)-f_0(\epsilon)\right] \end{aligned}$$

The term in the brackets under the integral decays as ϵ^{-2}, therefore the integral converges at $\epsilon \sim |\Delta|$ for large voltages and becomes independent of V for $eV \gg |\Delta|, T$. The current I_{exc} thus saturates for high voltages $V \to \infty$:

$$\begin{aligned} I_{exc}(\infty) &= \frac{1}{eR_N}\int_{-\infty}^{\infty} d\epsilon\ \left(\left[1+Z^2\right]\left[1-|b|^2+|a|^2\right]-1\right)\left[1-f_0(\epsilon)\right] \\ &= \frac{1}{2eR_N}\int_{-\infty}^{\infty} d\epsilon\ \left(\left[1+Z^2\right]\left[1-|b|^2+|a|^2\right]-1\right) \end{aligned}$$



Current through the NIS junction. NS Andreev current.

Consider zero barrier strength Z=0

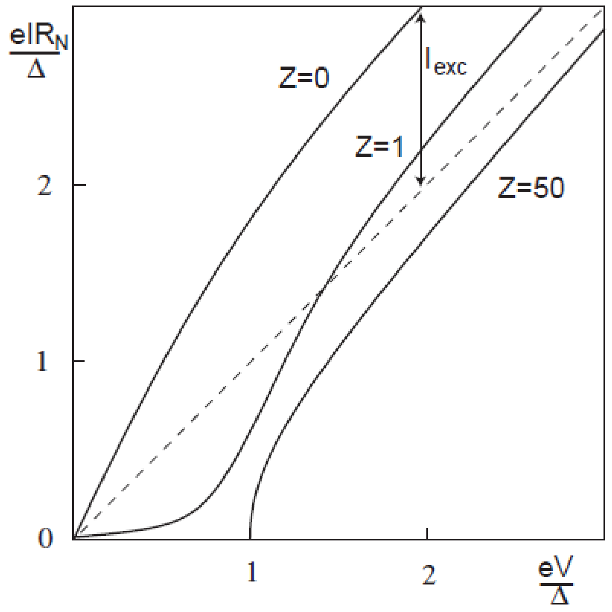
For $T \ll T_c$ and low voltages $eV \ll |\Delta|$ we need only $\epsilon \ll |\Delta$.

We have $|a|^2 = 1$ while $|b|^2 = 0$

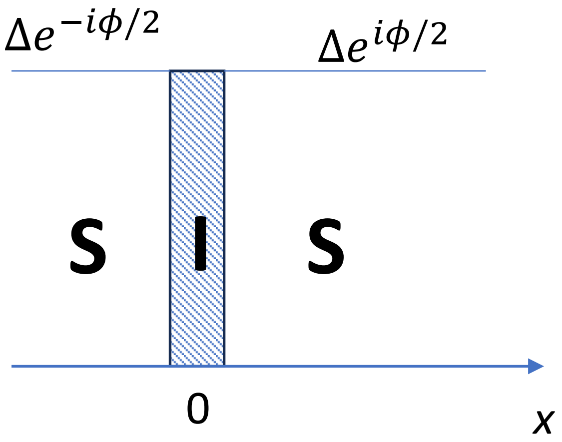
$$I_x = \frac{2}{eR_{\mathrm{Sh}}} \int_{-\infty}^{\infty} [f_0(\epsilon - eV) - f_0(\epsilon)]\, d\epsilon = \frac{2V}{R_{\mathrm{Sh}}}$$

The conductance is twice the normal-state conductance

This is due to the fact that the current through the interface
is carried by both electrons and holes.



SIS junction. Wave functions and the energy of bound states.



In the right superconductor:

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_2 e^{i\tilde{q}_+ x} \begin{pmatrix} \tilde{U} e^{i\phi/4} \\ \tilde{V} e^{-i\phi/4} \end{pmatrix} + d_2 e^{-i\tilde{q}_- x} \begin{pmatrix} \tilde{V} e^{i\phi/4} \\ \tilde{U} e^{-i\phi/4} \end{pmatrix}$$

$$q_\pm = k_x \pm i \frac{\sqrt{|\Delta|^2 - \epsilon^2}}{\hbar^2 k_x}$$

$$\tilde{U} = \frac{1}{\sqrt{2}} \left(1 + i \frac{\sqrt{|\Delta|^2 - \epsilon^2}}{\epsilon} \right)^{1/2}, \quad \tilde{V} = \frac{1}{\sqrt{2}} \left(1 - i \frac{\sqrt{|\Delta|^2 - \epsilon^2}}{\epsilon} \right)^{1/2}$$

In the left superconductor:

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 e^{-i\tilde{q}_+ x} \begin{pmatrix} \tilde{U} e^{-i\phi/4} \\ \tilde{V} e^{i\phi/4} \end{pmatrix} + d_1 e^{i\tilde{q}_- x} \begin{pmatrix} \tilde{V} e^{-i\phi/4} \\ \tilde{U} e^{i\phi/4} \end{pmatrix}$$

SIS junction. Wave functions and the energy of bound states.

Boundary conditions at *x=0*:

$$c_1\tilde{U}e^{-i\phi/4}+d_1\tilde{V}e^{-i\phi/4}=c_2\tilde{U}e^{i\phi/4}+d_2\tilde{V}e^{i\phi/4}$$
$$c_1\tilde{V}e^{i\phi/4}+d_1\tilde{U}e^{i\phi/4}=c_2\tilde{V}e^{-i\phi/4}+d_2\tilde{U}e^{-i\phi/4}$$

$$\left(c_2\tilde{U}e^{i\phi/4}-d_2\tilde{V}e^{i\phi/4}\right)-\left(-c_1\tilde{U}e^{-i\phi/4}+d_1\tilde{V}e^{-i\phi/4}\right)=-2iZ\left(c_2\tilde{U}e^{i\phi/4}+d_2\tilde{V}e^{i\phi/4}\right)$$
$$\left(c_2\tilde{V}e^{-i\phi/4}-d_2\tilde{U}e^{-i\phi/4}\right)-\left(-c_1\tilde{V}e^{i\phi/4}+d_1\tilde{U}e^{i\phi/4}\right)=-2iZ\left(c_2\tilde{V}e^{-i\phi/4}+d_2\tilde{U}e^{-i\phi/4}\right)$$

Excluding c_1 and d_1 we obtain:

$$c_2\tilde{U}\left[\left(\tilde{U}^2+iZ(\tilde{U}^2-\tilde{V}^2)\right)e^{i\phi/2}-\tilde{V}^2e^{-i\phi/2}\right]+d_2\tilde{V}\left[\left(\tilde{V}^2+iZ(\tilde{U}^2-\tilde{V}^2)\right)e^{i\phi/2}-\tilde{U}^2e^{-i\phi/2}\right]=0$$

$$c_2\tilde{V}\left[\left(\tilde{V}^2-iZ(\tilde{U}^2-\tilde{V}^2)\right)e^{-i\phi/2}-\tilde{U}^2e^{i\phi/2}\right]+d_2\tilde{U}\left[\left(\tilde{U}^2-iZ(\tilde{U}^2-\tilde{V}^2)\right)e^{-i\phi/2}-\tilde{V}^2e^{i\phi/2}\right]=0$$

Requiring zero of the determinant we find the condition of existence of a nonzero solution

$$4\tilde{U}^2\tilde{V}^2\cos^2(\phi/2)=1+Z^2\left(\tilde{U}^2-\tilde{V}^2\right)^2$$

SIS junction. Wave functions and the energy of bound states.

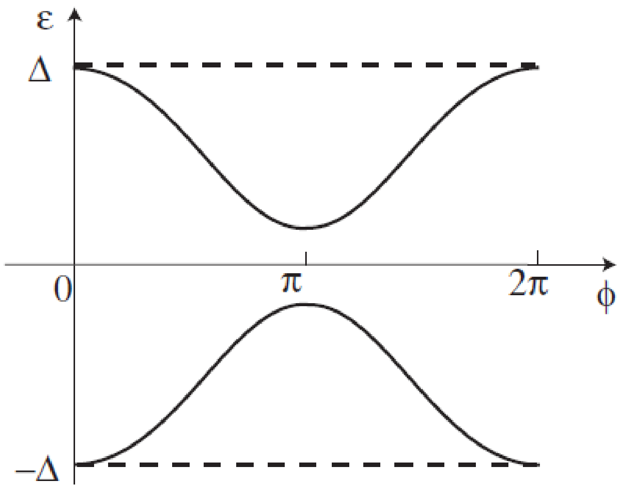
This yields $\epsilon = \pm\epsilon_\phi$ where

$$\epsilon_\phi = |\Delta|\sqrt{1 - \mathcal{T}\sin^2(\phi/2)}$$

where $\mathcal{T} = 1/(1+Z^2)$ is the transmission coefficient in the normal state

Without a barrier $Z = 0$ when $\mathcal{T} = 1$ we recover the spectrum of a ballistic point contact
For a final $\mathcal{T}$ the gap $|\Delta|\sqrt{1-\mathcal{T}} = |\Delta|\sqrt{\mathcal{R}}$ appears for $\phi = \pi$.

Energy spectrum of SIS junction



Supercurrent

Near the contact, $\lambda_s x \ll 1$

$$\begin{aligned} I &= \frac{2\hbar e}{m}\sum_n k_x \mathrm{Re}\left[f(\epsilon_n)\left(c_2^*\tilde{U}^* e^{-ik_x x} + d_2^*\tilde{V}^* e^{ik_x x}\right)\left(c_2\tilde{U}e^{ik_x x} - d_2\tilde{V}e^{-ik_x x}\right)\right. \\ &\quad \left. - [1 - f(\epsilon_n)]\left(c_2\tilde{V}e^{ik_x x} + d_2\tilde{U}e^{-ik_x x}\right)\left(c_2^*\tilde{V}^* e^{-ik_x x} - d_2^*\tilde{U}^* e^{ik_x x}\right)\right] \\ &= -\frac{2\hbar e}{m}\sum_n k_x[1 - 2f(\epsilon_n)]\left(|c_2|^2 - |d_2|^2\right)|\tilde{U}|^2 \end{aligned}$$

SIS junction. Supercurrent.

The final result for the supercurrent is:

$$\boxed{I = \frac{N_> \mathcal{T} e |\Delta|^2}{2\hbar} \frac{\sin\phi}{\epsilon_\phi} \tanh\left(\frac{\epsilon_\phi}{2T}\right)}$$

Here $N_>$ is the number of channels, $N_> = R_0/R_{\text{Sh}} = \dfrac{\pi\hbar}{e^2 R_{\text{Sh}}}$

The current can be also written as $$I = \frac{\pi|\Delta|^2}{2eR_N} \frac{\sin\phi}{\epsilon_\phi} \tanh\left(\frac{\epsilon_\phi}{2T}\right)$$

$\dfrac{1}{R_N} = \dfrac{\mathcal{T}}{R_{\text{Sh}}}$ is the conductance of a contact with a transparency $\mathcal{T}$.

For a fully transparent junction $\mathcal{T} = 1$ we recover the expression for the point contact. For a tunnel junction $\mathcal{T} \ll 1$ the current becomes: $I = I_c \sin\phi$

$$I_c = \frac{\pi|\Delta|}{2eR_N} \tanh\left(\frac{|\Delta|}{2T}\right)$$ - Ambegaokar and Baratoff result