Superconductivity. Phenomenological theory.

Lection 2 – London and GL models

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London theory.
Linear electrodynamics of superconductors

part 1

- **London equation. Robustness of the phase of the wave function**
- **London free energy.**
- **Examples. Thin film in parallel magnetic field.**
- **Image method.**
- **Short circuit principle.**
- **Kinetic inductance.**

London theory. Robustness of quantum mechanical phase.

$$J_{s0}(r) = -\frac{e\hbar}{2mi}\sum_{j=1}^{n_s}\int(\Psi_s^*\nabla_j\Psi_s - \Psi_s\nabla_j\Psi_s^*)\,\delta(r_j - r)\,d^3r_1 \ldots d^3r_{n_s}$$

$$J_s(r) = -\sum_{j=1}^{n_s}\frac{e^2}{mc}A(r)\int \Psi_s^*\Psi_s\,\delta(r_j - r)\,d^3r_1 \ldots d^3r_{n_s} = -\frac{n_s e^2}{mc}A(r),$$

$$\mathrm{rot}\,J_s = -\frac{n_s e^2}{mc}B$$

$$B(x) = B(0)\,e^{-\frac{x}{\lambda_L}}$$

$$\mathrm{rot}\,\mathrm{rot}\,B = \frac{4\pi}{c}\,\mathrm{rot}\,J_s$$

$$\Delta B = \frac{4\pi n_s e^2}{mc^2}B = \frac{1}{\lambda_L^2}B,$$

$$\lambda_L = \left(\frac{mc^2}{4\pi n_s e^2}\right)^{\frac{1}{2}}$$

London theory. Free energy functional

$$E_{kin} = n_s \frac{mv_s^2}{2} = \frac{mj_s^2}{2n_s e^2} = \frac{\lambda^2}{8\pi}\left(rot\vec{B}\right)^2 \qquad \lambda^2 = \frac{mc^2}{4\pi n_s e^2}$$

$$F_{sB} = F_{s0} + \frac{1}{8\pi}\int\left(\vec{B}^2 + \lambda^2\left(rot\vec{B}\right)^2\right)dV$$

$$rot\vec{B} = \frac{4\pi}{c}\left(-\frac{c}{4\pi\lambda^2}\vec{A}\right)$$

$$F_{sB} = F_{s0} + \frac{1}{8\pi}\int\left(\left(rot\vec{A}\right)^2 + \lambda^{-2}\vec{A}^2\right)dV$$

Variation of the free energy functional.

$$\delta F_{sB} = \frac{1}{4\pi}\int\left(\vec{B}\delta\vec{B} + \lambda^2 rot\vec{B}rot\,\delta\vec{B}\right)dV = \frac{1}{4\pi}\int\left(\vec{B}\delta\vec{B} + \lambda^2 rotrot\vec{B}\cdot\delta\vec{B}\right)dV$$

$$div\left[rot\vec{B}, \delta\vec{B}\right] = -rot\,\delta\vec{B}rot\vec{B} + rotrot\vec{B}\cdot\delta\vec{B}$$

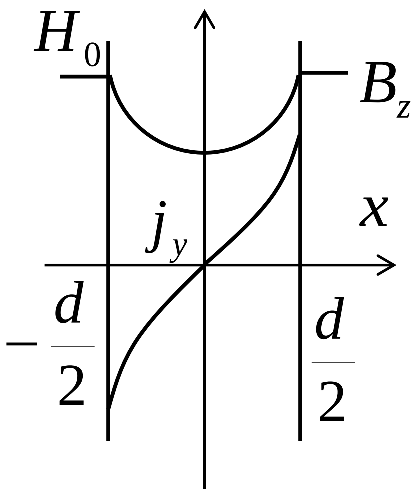
$$\vec{B} + \lambda^2 rotrot\vec{B} = 0$$

$$div\vec{A} = 0$$

$$\vec{A} + \lambda^2 rotrot\vec{A} = 0$$

$$\vec{n}\vec{A} = 0$$

Film in a parallel magnetic field.

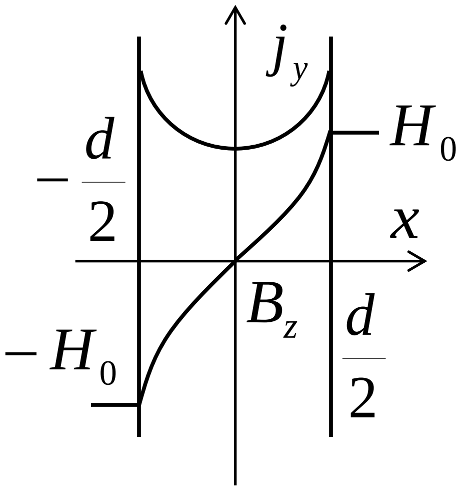


$$B - \lambda^2 B'' = 0$$

$$B = H_0 \frac{\cosh(x/\lambda)}{\cosh(d/2\lambda)}$$

$$j = -\frac{cH_0}{4\pi\lambda} \frac{\sinh(x/\lambda)}{\cosh(d/2\lambda)}$$

Film with current



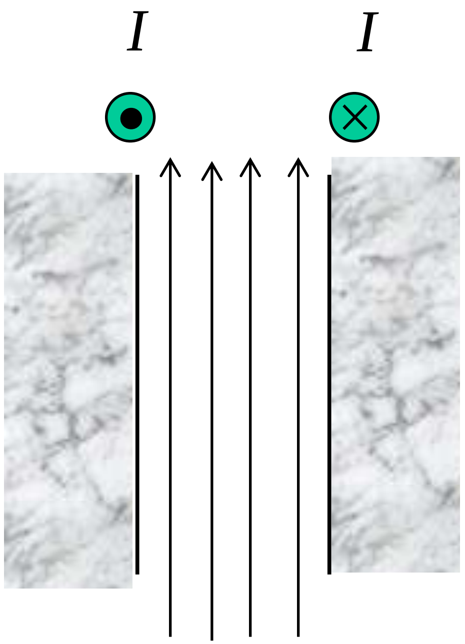
$$B - \lambda^2 B'' = 0$$

$$2\pi I / c = H_0$$

$$B = H_0 \frac{\sinh(x/\lambda)}{\sinh(d/2\lambda)}$$

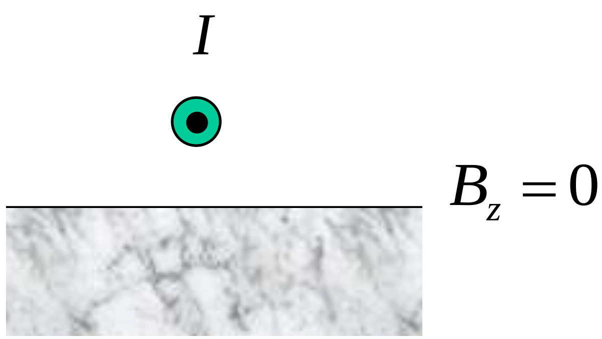
$$j = -\frac{cH_0}{4\pi\lambda} \frac{\cosh(x/\lambda)}{\cosh(d/2\lambda)}$$

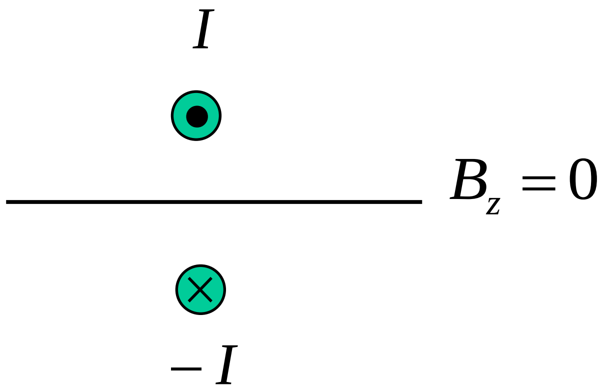
Two films



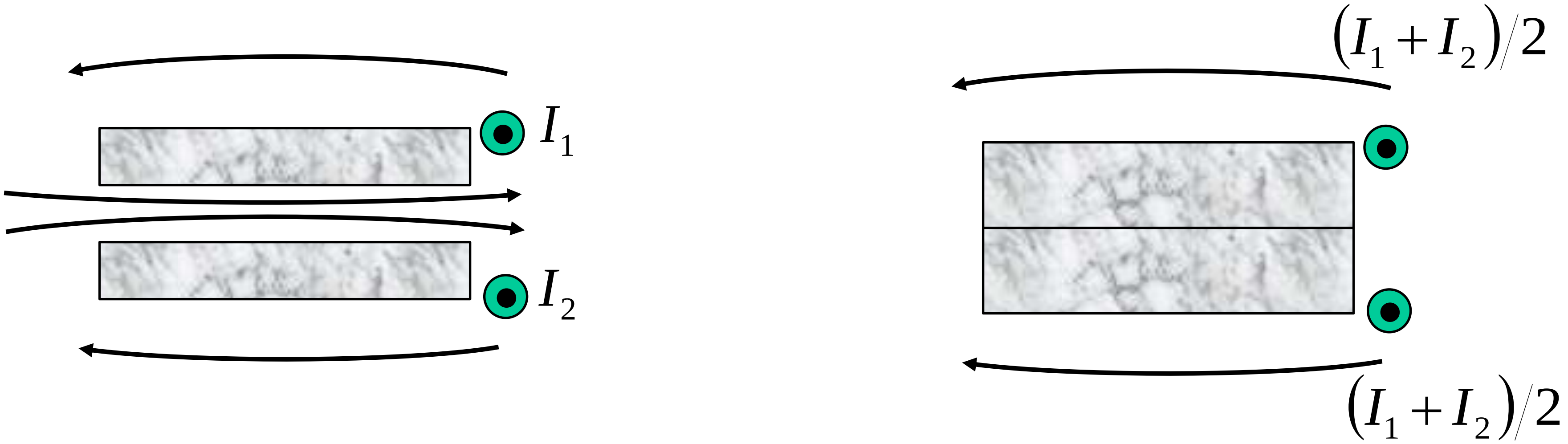
$$4\pi I / c = H_0$$

Image method





Short circuit principle



Current at internal side = $I_1 - (I_1 + I_2)/2 = (I_1 - I_2)/2$

Kinetic inductance

Geometrical inductance

$$F = \frac{1}{8\pi}\int \vec{B}^2 dV = \frac{1}{2c^2} L_{геом} I^2$$

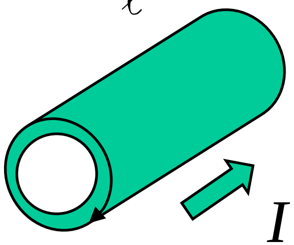
Kinetic inductance

$$F = \int \frac{nmv^2}{2} dV = \frac{1}{2c^2} L_{кин} I^2$$

$$L_{кин} = \frac{4\pi\lambda^2}{I^2}\int j_s^2 dV$$

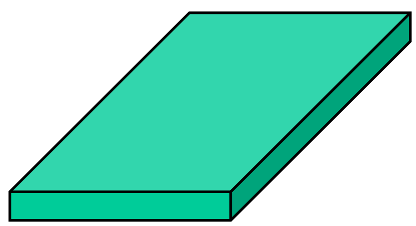
ℓ

$R >> \lambda$



$$I = 2\pi R\lambda j_s$$

$$L_{кин} = \frac{2\ell\lambda}{R}$$



$d << \lambda$

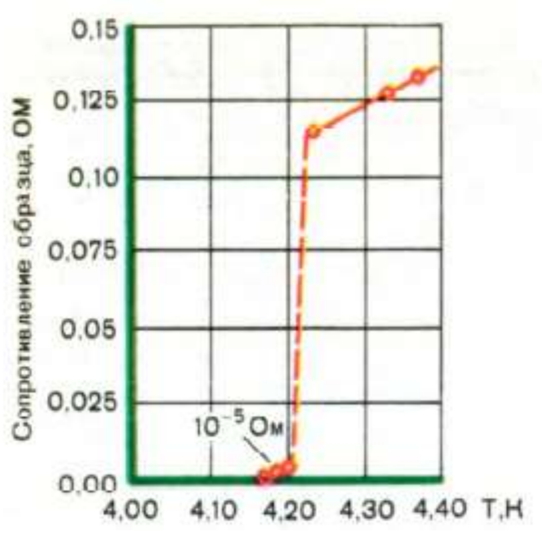
$$L_{кин} \propto \frac{\lambda^2}{d}$$

Ginzburg- Landau theory

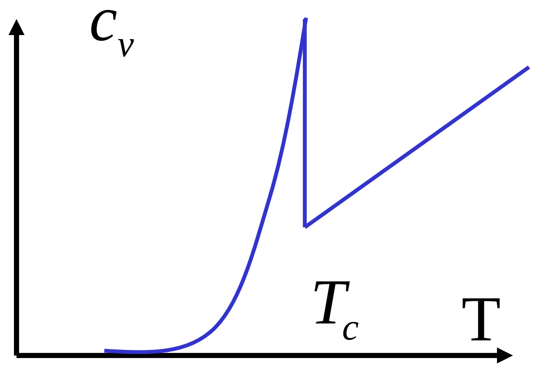
part 2

- **Phenomenological Landau theory for 2nd order phase transition. Order parameter. Examples.**
- **Equations for order parameter.**
- **order parameter for superconducting electrons.**
- **Ginzburg – Landau equations.**
- **Boundary conditions.**

Which properties of superconductors should we take account of?



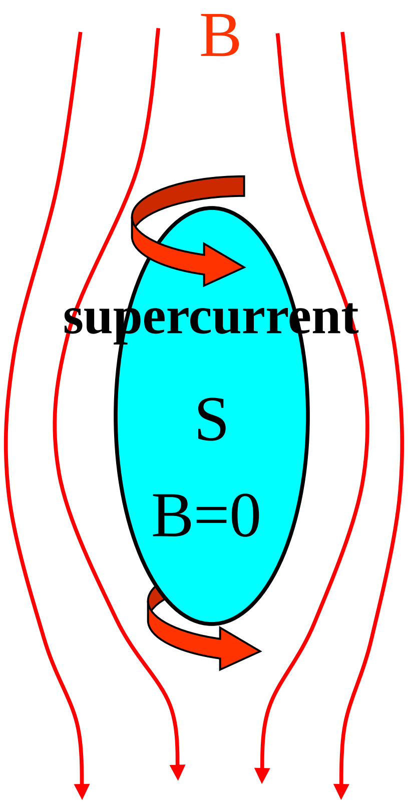
Dissipationless current (no resistance)

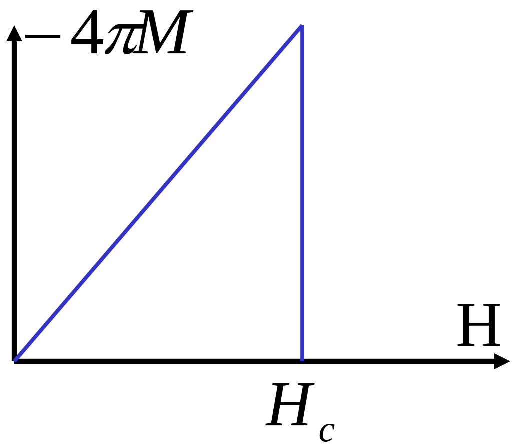


Jump of heat capacity

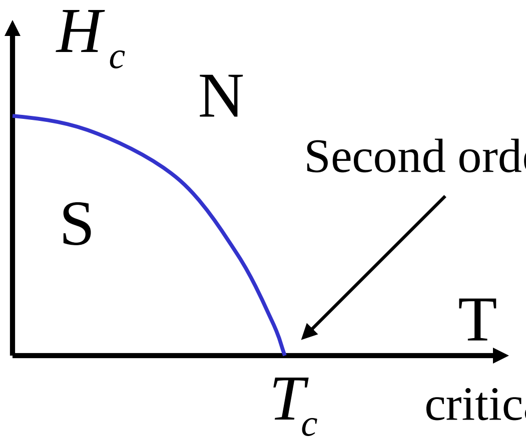
Which properties of superconductors should we take account of?

Meissner effect





1st order phase transition in magnetic field

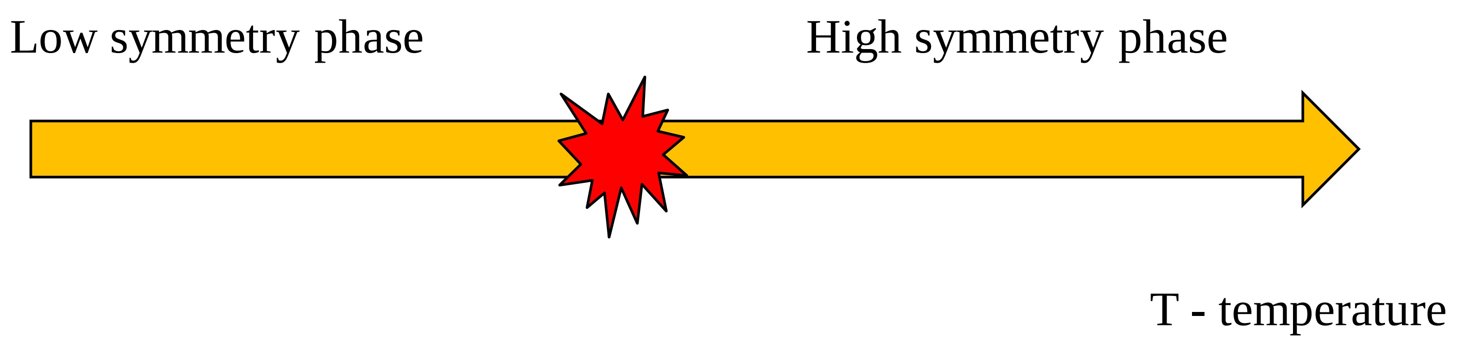


Second order phase transition at zero field

critical temperture of superconducting transition

How to develop description of the phase transition?
What is the key characteristic of the phase transition?
Symmetry breaking at the 2nd order phase transition.
Landau theory

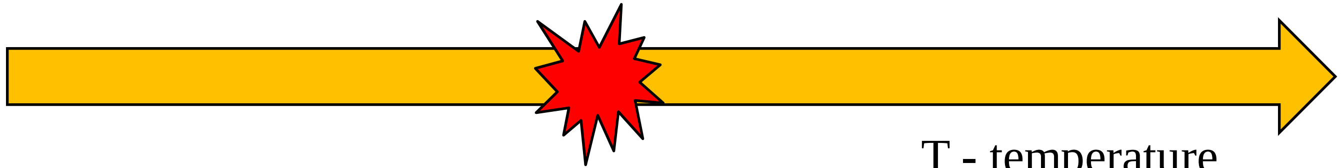
Second order phase transitions – phase transitions for which the second derivatives of thermodynamic potentials over pressure and temperature experience discontinuities (jumps)
While first derivatives change smoothly.



examples

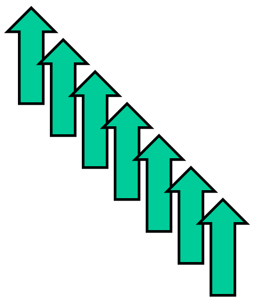
Low symmetry phase

High symmetry phase

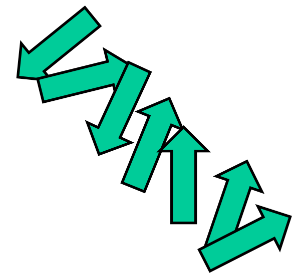


T - temperature

ferromagnetic

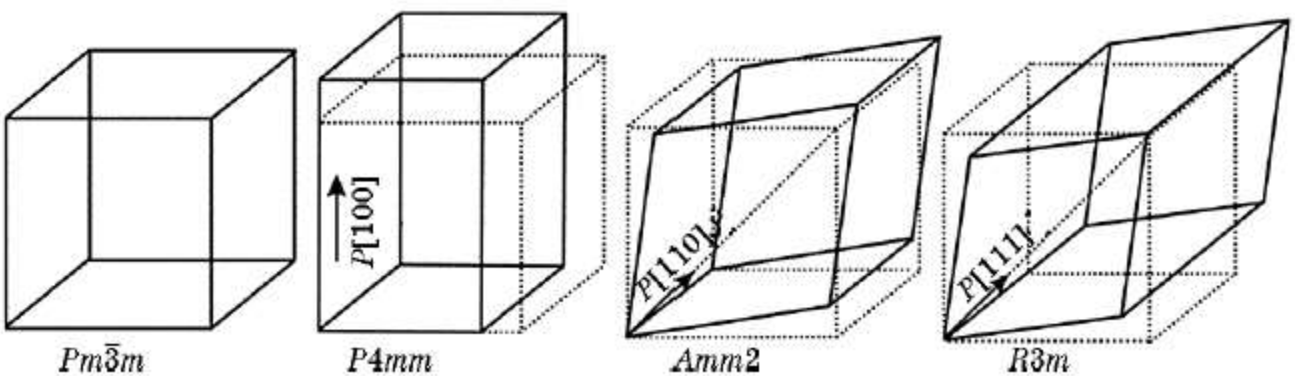


paramagnetic



Rotational symmetry breaking:
Appearance of preferable
direction

Structural transitions. Breaking of crystal synnetry



Order parameter

ferromagnetic: $\vec{M}$

Structural transition: $\vec{d}$

Superconducting transition?

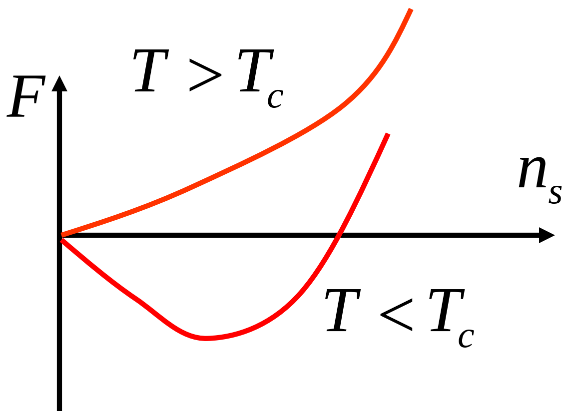
Concentration of superconducting electrons?

n_s

Landau theory: expansion of the free energy in powers of the order parameter

$$F = F_n + \int \left(a n_s + b n_s^2 \right) dV$$

$$a = \alpha (T - T_c)$$



Problem: how to include the magnetic field in our theory?

Ginzburg – Landau theory

Order parameter $$\Psi = \sqrt{\frac{n_s}{2}} e^{i\varphi}$$

Density of superconducting electrons

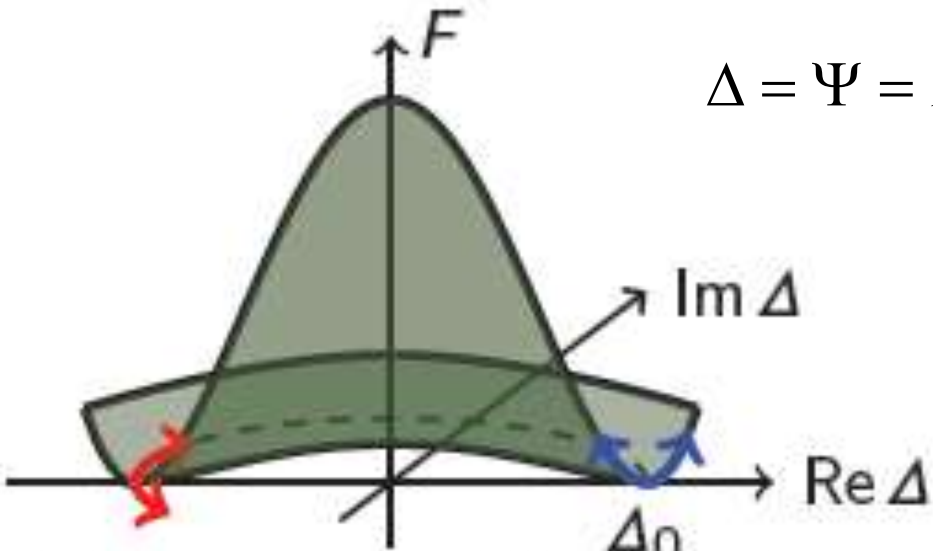
For slow changes of the order parameter in space:

Free energy $$F = F_n + \int \left(\frac{\hbar^2}{4m} \left|\nabla \Psi\right|^2 + a\left|\Psi\right|^2 + \frac{b}{2}\left|\Psi\right|^4 \right) dV$$

Free energy of the normal state

$$a = \alpha(T - T_c)$$

Equilibrium order parameter in homogeneous superconductor $$\left|\Psi\right|^2 = \frac{\alpha(T_c - T)}{b}$$



$$\Delta = \Psi = \sqrt{\frac{n_s}{2}} e^{i\varphi}$$

Let us now switch on the magnetic field:

Gauge invariance

$$\nabla\Psi \to \nabla\Psi - \frac{2ie}{\hbar c}\vec{A}\Psi$$

?

$$\tilde{F} = F_n + \int\left(\frac{\hbar^2}{4m}\left|\left(\nabla - \frac{2ie}{\hbar c}\vec{A}\right)\Psi\right|^2 + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{\vec{B}^2}{8\pi} - \frac{\vec{B}\vec{H}}{4\pi}\right)dV$$

Energy of magnetic field

In the external field $\vec{H}$

Variation of the functional

Over Ψ^* → **Ginzburg – Landau equations**

Over $\vec{A}$ → **Maxwell equations**

Q. What symmetry breaks down at superconducting transition?
A. Gauge symmetry

Q. Why we do not include odd powers of the psi-function and higher order derivatives ?
A. Symmetry arguments + weak nonlocal effects

Q. How to choose the signes of the coefficients?
A. Stability of the ground state

$$-\frac{\hbar^2}{4m}\left(\nabla-\frac{2ie}{\hbar c}\vec{A}\right)^2\Psi+a\Psi+b|\Psi|^2\Psi=0$$

$$rot\vec{B}=\frac{4\pi}{c}\vec{j}$$

$$\vec{j}=-\frac{ie\hbar}{2m}\left(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*\right)-\frac{2e^2}{mc}|\Psi|^2\vec{A}$$

$$\vec{j}=\frac{e\hbar}{m}|\Psi|^2\left(-\frac{2e}{\hbar c}\vec{A}+\nabla\varphi\right)$$

$$\vec{j}=-\frac{e^2 n_s}{mc}\vec{A}$$

London relation

Boundary conditions

I-S

$$\vec{n}\left(\nabla\Psi-\frac{2ie}{\hbar c}\vec{A}\Psi\right)=0$$

N-S

$$\Psi=0$$

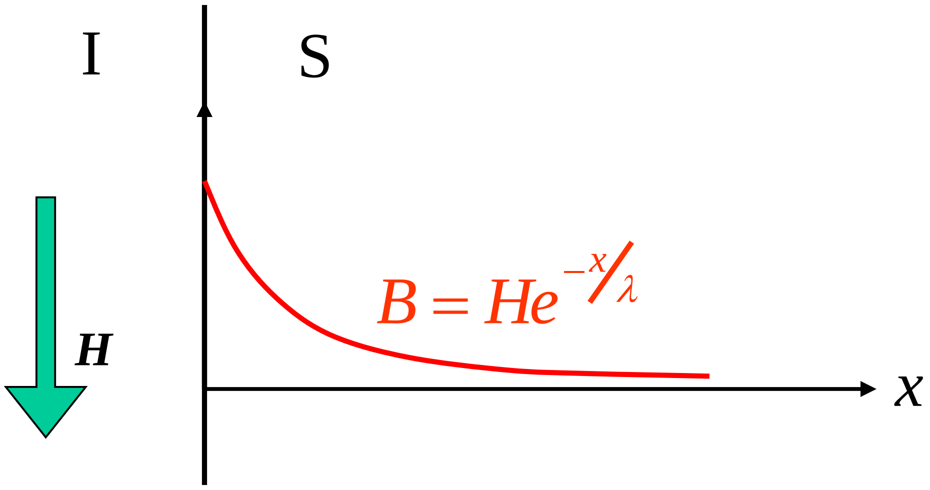
Superfluid velocity

$$\vec{j} = en_s\vec{V}_s = en_s \frac{\hbar}{2m}\left(\nabla\varphi - \frac{2e}{\hbar c}\vec{A}\right)$$

Characteristic lengths in Ginzburg – Landau theory

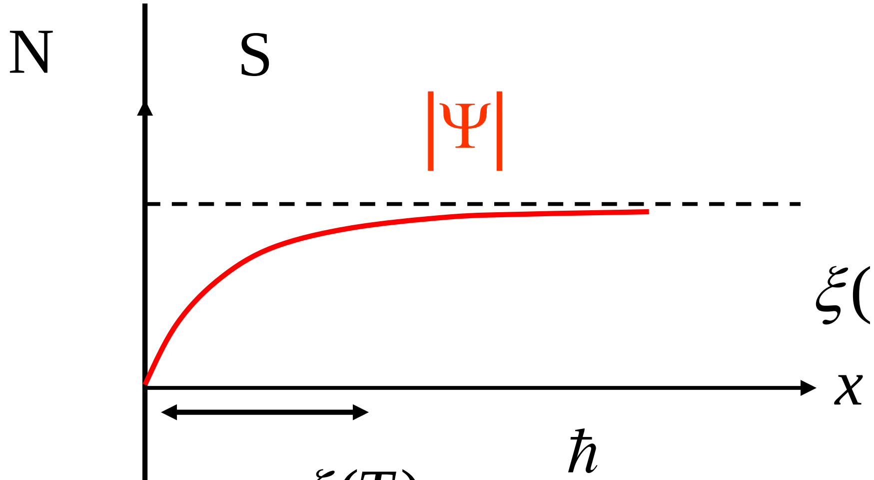
London penetration depth

$$\lambda(T) = \sqrt{\frac{mc^2 b}{8\pi e^2 |a|}} = \sqrt{\frac{mc^2}{4\pi e^2 n_s}}$$



$$\lambda(0) \sim 10^3\,\mathring{A}$$

Correlation length = characteristic length scale of the order parameter modulation



$$\xi(0) \sim 10 - 10^4\,\mathring{A}$$

$$\sim \xi(T) = \frac{\hbar}{2\sqrt{m|a|}}$$

Ginzburg – Landau parameter = $\kappa = \dfrac{\lambda(T)}{\xi(T)}$

Thermodynamic critical field. Flux quantum.

$$|\Psi|^2 = -\frac{a}{b} = \frac{\alpha(T_c - T)}{b}$$

$$F_s - F_n = -V\,\frac{\alpha^2 (T_c - T)^2}{2b} = -V\,\frac{H_{cm}^2}{8\pi}$$

$$H_{cm} = \left(\frac{4\pi\alpha^2}{b}\right)^{1/2} (T_c - T) = \frac{\phi_0}{2\pi\sqrt{2}\lambda\xi}$$

$$\phi_0 = \frac{\pi\hbar c}{e} = 2\cdot 10^{-7}\, Гс \cdot см^2$$

Generalization of the GL theory

Superconductors with anisotropic properties or multicomponent superconductors

$\Psi =$ **Vector or tensor**

Instabilities and nonlocal corrections

Layered superconductors. HTSC

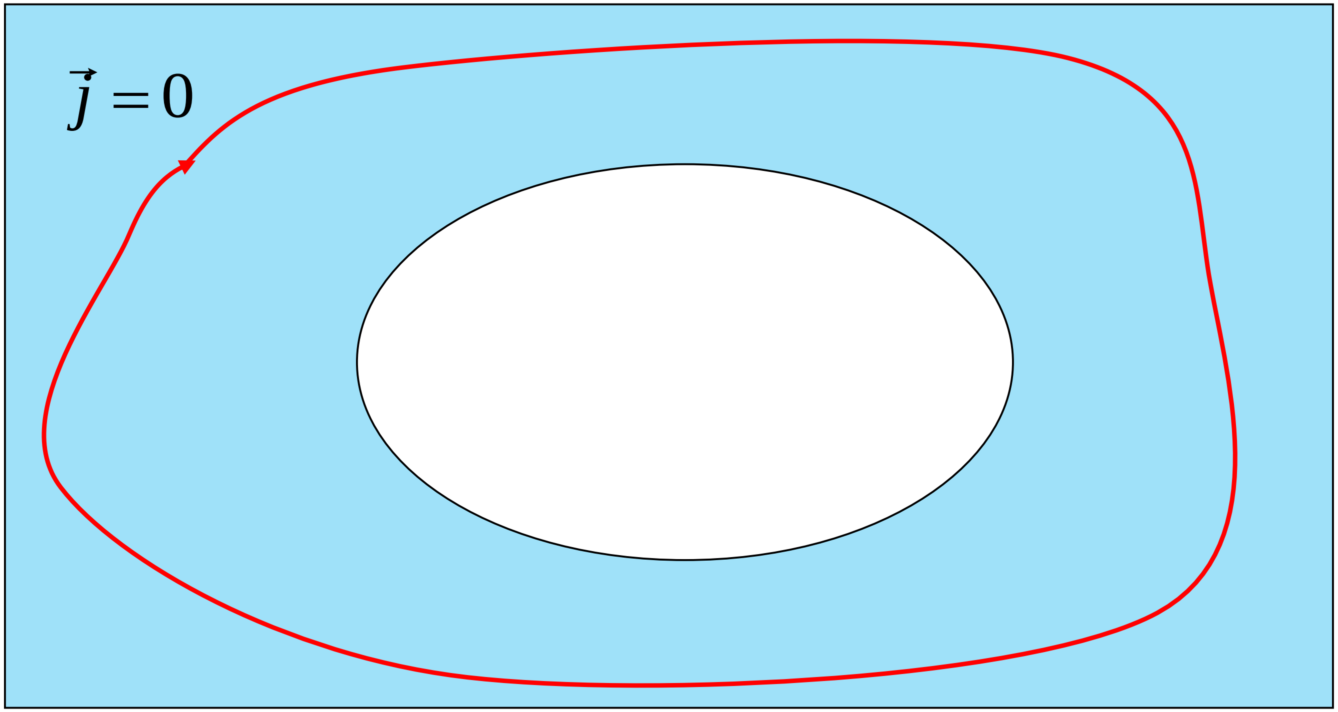
Magnetic flux quantization

$$\vec{j} = \frac{e\hbar}{m}|\Psi|^2\left(-\frac{2e}{\hbar c}\vec{A} + \nabla\varphi\right)$$

$$-\frac{2e}{\hbar c}\oint \vec{A}d\vec{\ell} + 2\pi n = 0$$

$$-\frac{2e}{\hbar c}\Phi + 2\pi n = 0$$

$$\Phi = n\phi_0$$



Ginzburg- Landau Zoo.

problems and solutions…

part 3

- **general problem to find critical temperature of superconducting instability**

- **critical fields and currents for a film**

- **energy of superconducting – normal phase boundary**

Problems of calculation of critical temperature of superconductivity nucleation. How to solve them? Upper critical field.

$$-\frac{\hbar^2}{4m}\left(\nabla - \frac{2ie}{\hbar c}\vec{A}\right)^2 \Psi = -a\Psi$$

$$\vec{A} = -\vec{x}_0 H_z y$$

energy E

Min E ⇒ ***Max critical T (H)***

$$E_0 = \frac{e\hbar H}{2mc} \quad \Rightarrow \quad H_{c2} = \frac{\phi_0}{2\pi\xi^2}$$

Superconducting nucleus

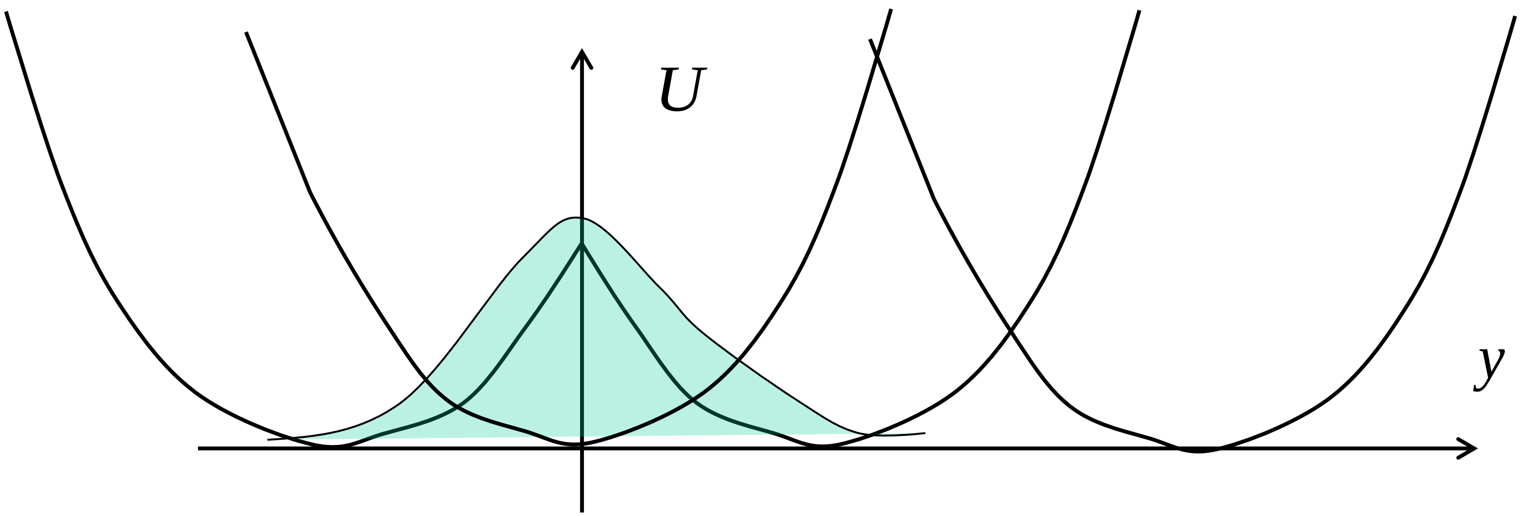
$$\Psi = A\exp\left(ikx - \frac{(y - kL_H^2)^2}{2L_H^2}\right)$$

Magnetic length $L_H = \sqrt{\frac{\hbar c}{2eH}}$

Superconductivity nucleation. Third critical field.

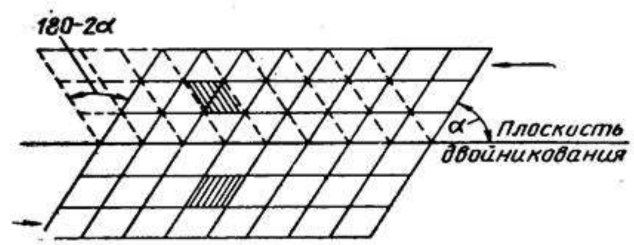
$$-\frac{\hbar^2}{4m}\left(\nabla - \frac{2ie}{\hbar c}\vec{A}\right)^2 \Psi = -a\Psi$$

$$\vec{A} = -\vec{x}_0 H_z y \qquad \frac{\partial \Psi}{\partial y} = 0 \qquad H_{c3} = 1.69 H_{c2}$$

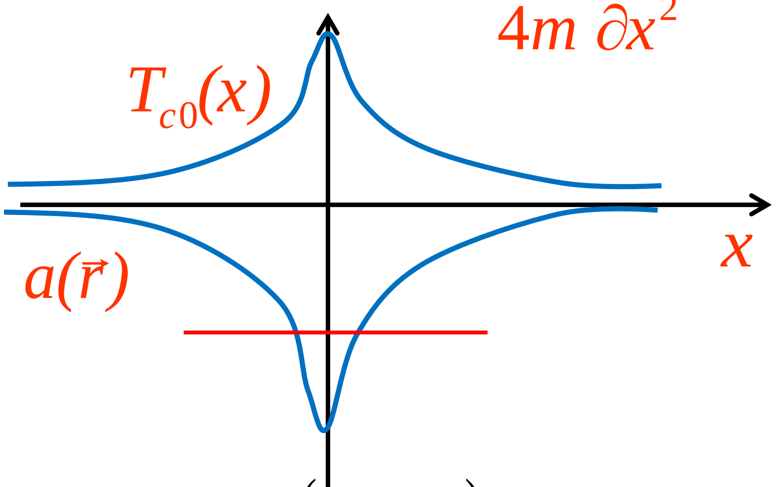


$$\Psi = A(y)\exp(ikx)$$

Superconductivity nucleation. Twinning planes.



$$-\frac{\hbar^2}{4m}\frac{\partial^2}{\partial x^2}\Psi + a(\vec{r})\Psi = 0$$



$$a = \alpha(T - T_c(x)) = \alpha(T - T_{c0} - \tau d\delta(x))$$

$$-\frac{\hbar^2}{4m}\left[\Psi'\right]_{x=0} - \alpha\tau d\Psi_{x=0} = 0$$

$$\Psi = A\exp\left(-|x|/\ell\right)$$

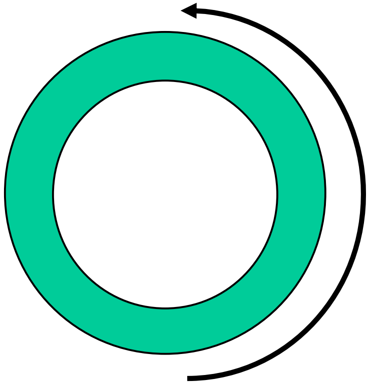
$$\hbar^2\ell^{-1} = 2m\alpha\tau d$$

$$\hbar^2\ell^{-2} = 4m\alpha(T - T_{c0})$$

$$\frac{m\alpha\tau^2 d^2}{\hbar^2} = T - T_{c0}$$

Superconductivity nucleation. Little – Parks effect

Superconducting thin-wall cylinder



W.A. Little and R.D. Parks (1962,1964)

$$A_\varphi = \frac{Hr}{2} \qquad \Psi = |\Psi| e^{i\chi}$$

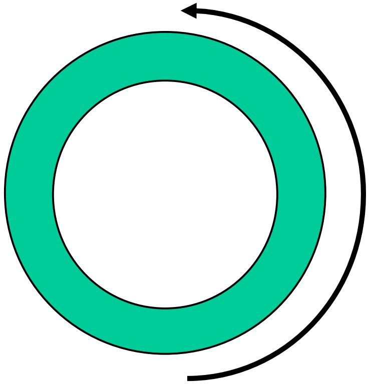
$$F = F_n + \int \left(\frac{\hbar^2}{4m} \left(\nabla\chi - \frac{2e}{\hbar c} \vec{A} \right)^2 |\Psi|^2 + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 \right) dV$$

$$\Psi = |\Psi| e^{iL\theta}$$

$$F = F_n + \int \left(\frac{\hbar^2}{4mR^2} \left(L - \frac{eH\pi R^2}{\pi\hbar c} \right)^2 |\Psi|^2 + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 \right) dV$$

Superconductivity nucleation. Little – Parks effect

Superconducting thin-wall cylinder



W.A. Little and R.D. Parks (1962,1964)

$$\Psi = |\Psi| e^{iL\theta}$$

$$A_\varphi = \frac{Hr}{2}$$

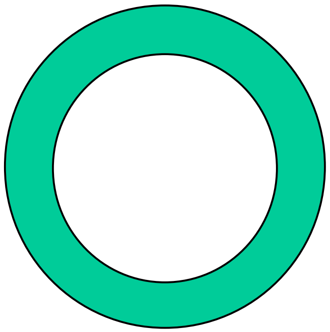
$$\Psi = |\Psi| e^{i\chi}$$

$$F = F_n + \int \left(\frac{\hbar^2}{4mR^2} \left(L - \frac{\Phi}{\phi_0} \right)^2 |\Psi|^2 + a|\Psi|^2 + \frac{b}{2} |\Psi|^4 \right) dV$$

$$F = F_n + \int \left(\left(\frac{\hbar^2}{4mR^2} \left(L - \frac{\Phi}{\phi_0} \right)^2 + a \right) |\Psi|^2 + \frac{b}{2} |\Psi|^4 \right) dV$$

Superconductivity nucleation. Little – Parks effect

Superconducting thin-wall cylinder



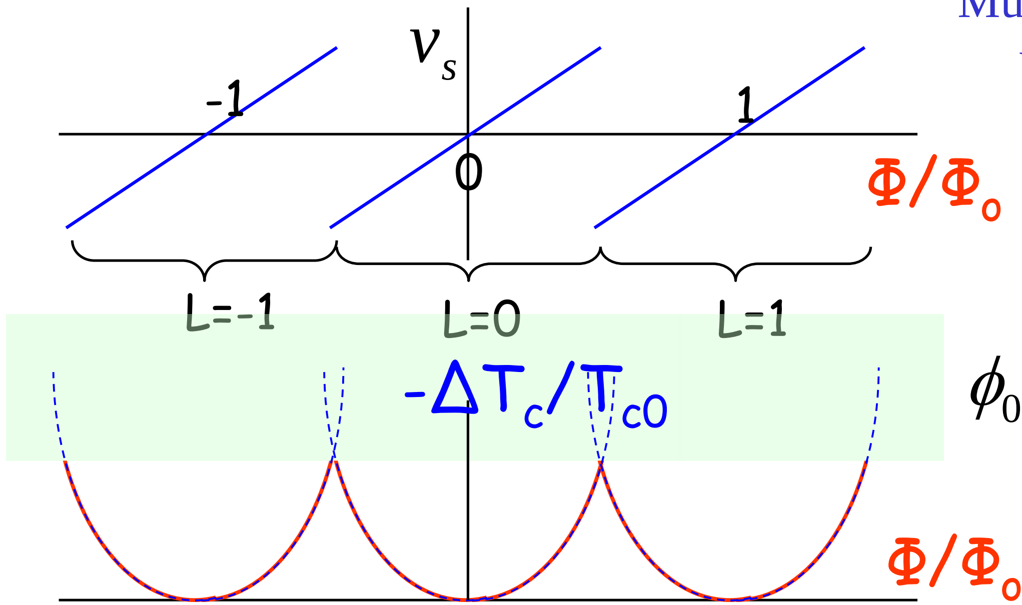
$$v_s = \frac{\hbar}{mR}\left(L - \frac{\Phi}{\Phi_0}\right)$$

$$\frac{\Delta T_c(H)}{T_{c0}} \sim \left(\frac{m v_s}{\hbar}\right)^2$$

W.A. Little and R.D. Parks (1962,1964)

T_c (H) oscillations

Multiquantum vortices



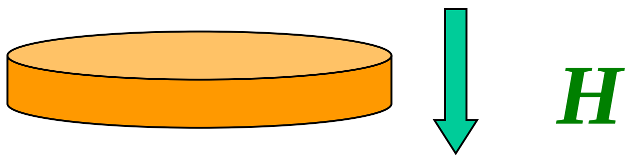
$$\phi_0 = \frac{\pi\hbar c}{e}$$

Little-Parks effect

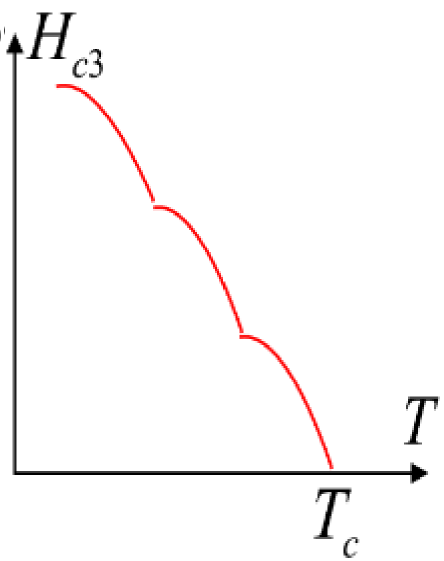
Simply-connected systems

T_c (H) oscillations

Mesoscopic samples
dimensions ~ several coherence lengths



O.Buisson et al (1990)
R.Benoist, W.Zwerger (1997)
V.A.Schweigert, F.M.Peeters (1998)
H.T.Jadallah, J.Rubi
Sternberg (199



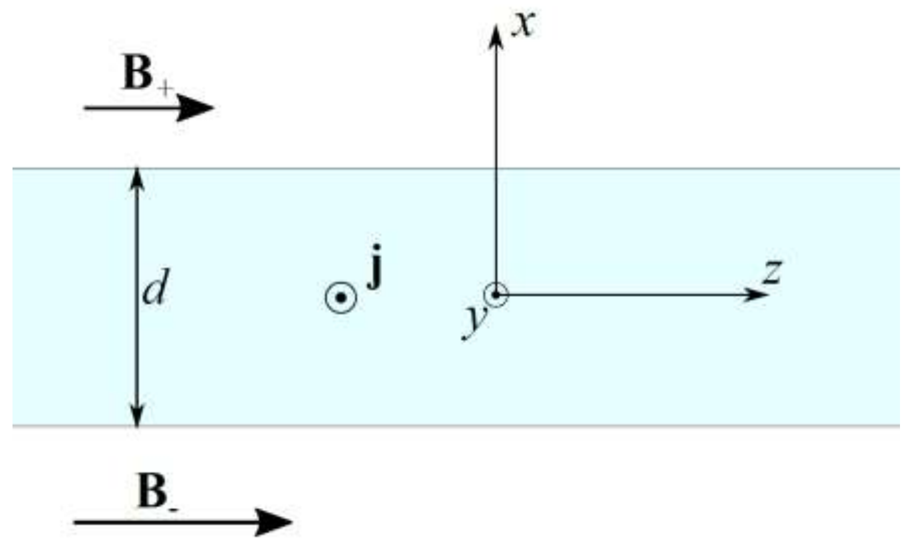
Multiquantum vortices

Multiquantum vortices around magnetic dots

I.K. Marmorkos, A. Matulis, F.M. Peeters (1996)
S.-L. Cheng, H.A. Fertig (1999)
M.V. Milosevic, S.V.Yampolskii, F.M. Peeters (2002)
M.V. Milosevic, F.M. Peeters (2003)

Origin of T_c oscillations: Transitions between the states with different vorticity L

Critical fields and currents in thin films



$$\frac{d^2A_y}{dx^2} = \frac{16\pi e^2}{mc^2}\psi^2 A_y$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{2e^2}{mc^2}A_y^2\psi - |a_{GL}|\,\psi + b_{GL}\psi^3 = 0$$

$$\frac{dA_y}{dx}\left(\frac{d}{2}\right) = B_+ \qquad \frac{dA_y}{dx}\left(-\frac{d}{2}\right) = B_-$$

$$\frac{d\psi}{dx}\left(\pm\frac{d}{2}\right) = 0$$

Dimensionless equations

Units of ψ – $\sqrt{\frac{|a|_{GL}}{b_{GL}}}$

Units of magnetic field – $\frac{\phi_0}{2\pi\lambda\xi}$

Единицы длины – λ

Units of vector potential – $\frac{\phi_0}{2\pi\xi}$

$$\frac{d^2 A}{dx^2} = \psi^2 A$$ (3A)

$$-\varkappa^{-2}\frac{d^2\psi}{dx^2} + A^2\psi - \psi + \psi^3 = 0$$ (3B)

$$\frac{dA_y}{dx}\left(\frac{d}{2}\right) = B_+ \qquad \frac{dA_y}{dx}\left(-\frac{d}{2}\right) = B_-$$

$$\frac{d\psi}{dx}\left(\pm\frac{d}{2}\right) = 0$$

Critical field of thin film

Thin film: $d \ll 1, \varkappa$.

$$\psi(x) = \psi_0 + \psi_1(x)$$

$$\frac{d^2A}{dx^2} = \psi_0^2 A \quad \Rightarrow \quad A(x) = A_1 e^{-x/\psi_0} + A_2 e^{x/\psi_0}$$

$$-\varkappa^{-2}\frac{d^2\psi_1}{dx^2} + A^2\psi_0 - \psi_0 + \psi_0^3 = 0 \qquad \frac{d\psi_1}{dx}\left(\pm\frac{d}{2}\right) = 0$$

Solvability condition:

$$\int_{-d/2}^{d/2}\left[A^2(x)\psi_0 - \psi_0 + \psi_0^3\right]dx = 0$$

For problem of critical field $B_+ = B_-$,

$$A = \frac{B_+ \sinh(\psi_0 x)}{\psi_0 \cosh\left(\psi_0 \frac{d}{2}\right)}$$

$$B_+^2 = \frac{12(1-\psi_0^2)}{d^2} \quad \Rightarrow \quad B_c = \frac{\sqrt{12}}{d}, \quad \text{или} \quad B_c = \frac{\phi_0\sqrt{3}}{\pi d\xi}$$

Critical current of thin film

$B_+ = -B_-$, $$A = \frac{B_+ \cosh(\psi_0 x)}{\psi_0 \sinh\left(\psi_0 \frac{d}{2}\right)} \approx \text{const}$$

$$A = \sqrt{1 - \psi_0^2}$$

$$j \propto \psi_0^2 A = \psi_0^2 \sqrt{1 - \psi_0^2}$$

$\psi_0^2 = 2/3$ –this value gives the maximum current.

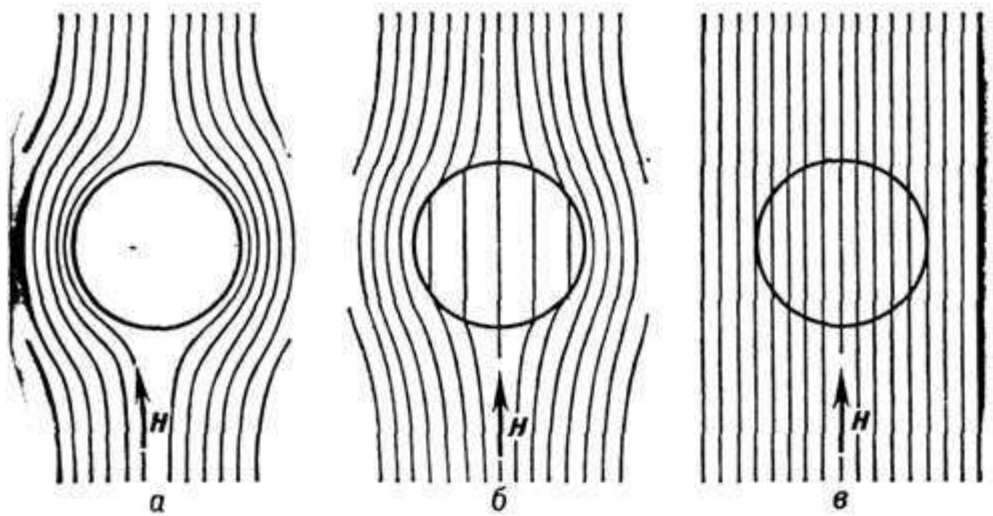
Critical current density

$$\boxed{j_d = \frac{c\,|\phi_0|}{12\pi^2 \lambda^2 \xi \sqrt{3}}}$$

Intermediate state

$$H < H_{cm}$$

$$H_{cm}$$



Energy of SN boundary

$$\frac{d^2A}{dx^2} = \psi^2 A \quad \text{(3A)}$$

$$-\varkappa^{-2}\frac{d^2\psi}{dx^2} + A^2\psi - \psi + \psi^3 = 0 \quad \text{(3B)}$$

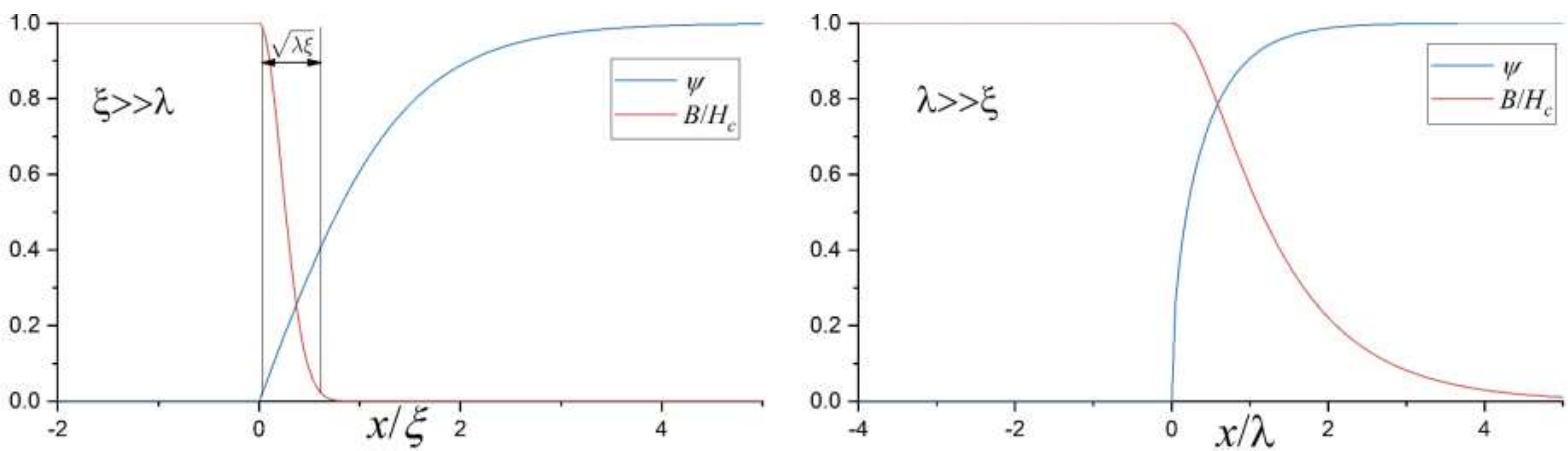
$$\frac{dA}{dx}(-\infty) = \frac{1}{\sqrt{2}} \qquad \psi(-\infty) = 0$$

$$A(+\infty) = 0 \qquad \psi(+\infty) = 1$$

$$\tilde{F} = F_{GL} + \int \frac{(\mathbf{B} - \mathbf{H}_e)^2}{8\pi} d^3\mathbf{r}$$

$$H_e = H_c \qquad \frac{\tilde{F}}{S} = \frac{H_c^2\lambda}{4\pi}\sigma$$

$$\sigma = ?$$



Magnetic field and order parameter close to the boundary

$$\sigma = \int_{-\infty}^{\infty} \left[\varkappa^{-2} \left(\frac{d\psi}{dx} \right)^2 + A^2\psi^2 - \psi^2 + \frac{\psi^4}{2} + \left(\frac{dA}{dx} - \frac{1}{\sqrt{2}} \right)^2 \right] dx$$

$$\sigma = \int_{-\infty}^{\infty} \left[-\frac{\psi^4}{2} + \left(\frac{dA}{dx} - \frac{1}{\sqrt{2}} \right)^2 \right] dx$$

$$\frac{\partial \sigma}{\partial \varkappa} = -2\varkappa^{-3} \int_{-\infty}^{\infty} \left(\frac{d\psi}{dx} \right)^2 dx < 0$$

$$-\varkappa^{-2}\left(\frac{d\psi}{dx}\right)^2 + A^2\psi^2 - \left(\frac{dA}{dx}\right)^2 + \frac{(1-\psi^2)^2}{2} = 0$$

$$\sigma = \int_{-\infty}^{\infty}\left[\varkappa^{-2}\left(\frac{d\psi}{dx}\right)^2 + A^2\psi^2 - \psi^2 + \frac{\psi^4}{2} + \left(\frac{dA}{dx} - \frac{1}{\sqrt{2}}\right)^2\right]dx$$

$$\sigma = 2\int_{-\infty}^{\infty}\left[\frac{1}{\kappa^2}\left(\frac{d\psi}{dx}\right)^2 + \frac{dA}{dx}\left(\frac{dA}{dx} - \frac{1}{\sqrt{2}}\right)\right]dx$$

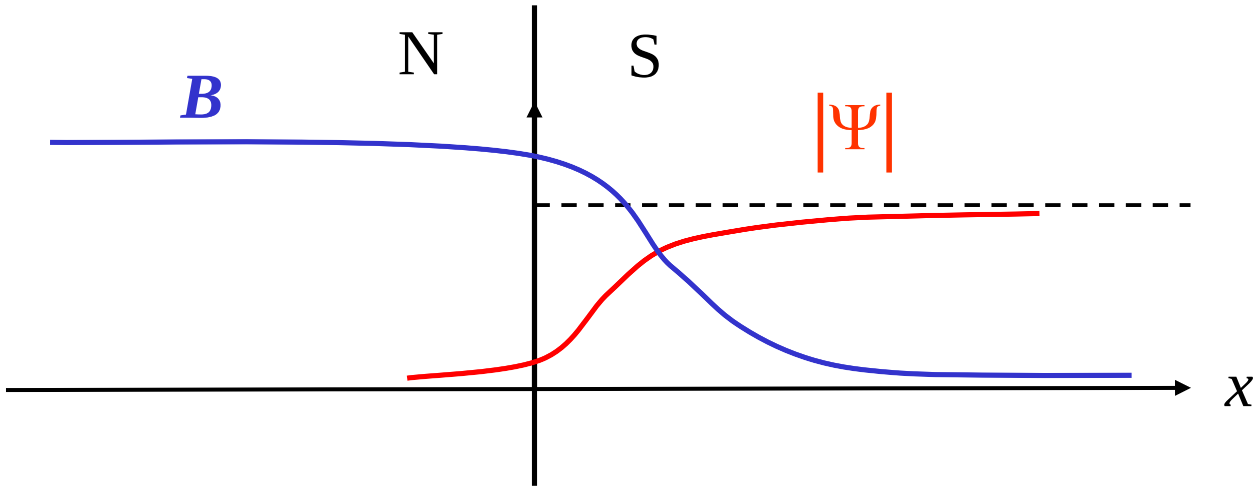
$\varkappa >> 1$

$$\sigma = -\frac{4(\sqrt{2}-1)}{3}$$

$\varkappa << 1$

$$\sigma = \frac{2\sqrt{2}}{3\varkappa}$$

Energy of N-S boundary



$$\kappa < \frac{1}{\sqrt{2}}$$

Boundary energy > 0

$$\kappa > \frac{1}{\sqrt{2}}$$

Boundary energy < 0