Superconductivity. Phenomenological theory. Lection 2 – Type-II superconductivity. Vortex matter

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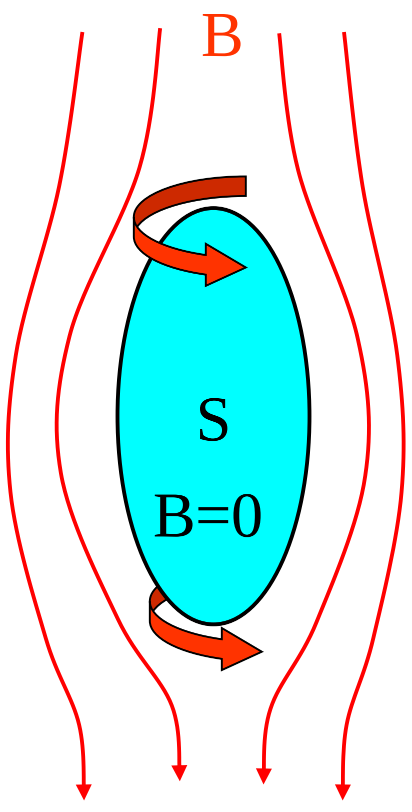
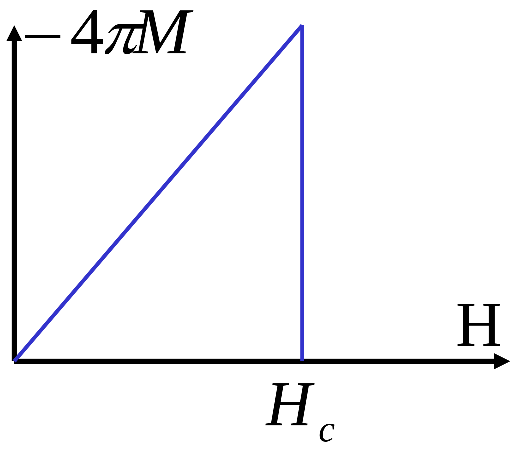
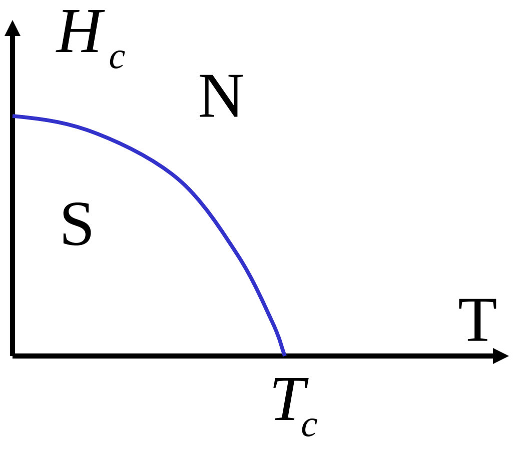
outline

- **type-I and type –II superconductors**
- **Isolated vortex.**
- **vortex lattice.**
- **properties of vortex state.**

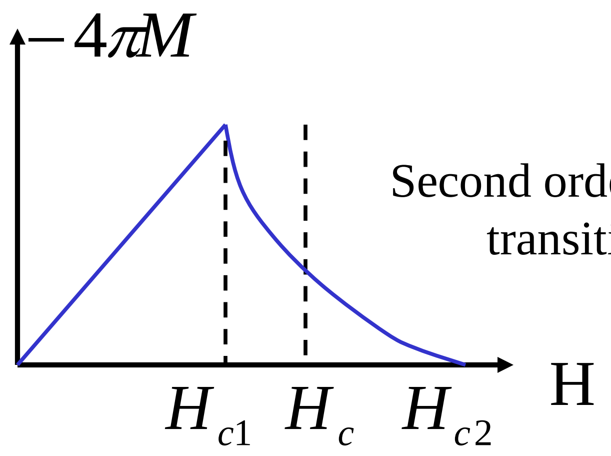
Type-I superconductors.



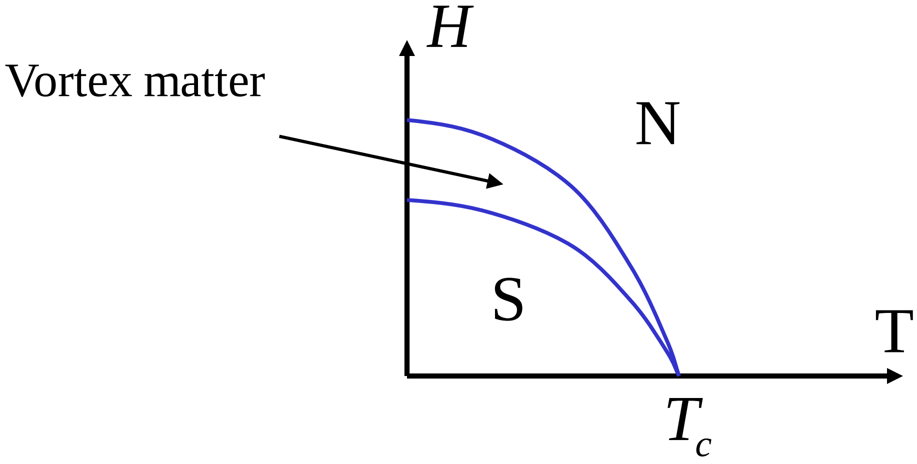




Type-II superocnductivity



Second order phase transition



$\kappa > \frac{1}{\sqrt{2}}$ → *SN boundaries are energetically favorable*

What do we call "vortices" and why do need them?

Partial penetration of magnetic field into superconductor →

London equation

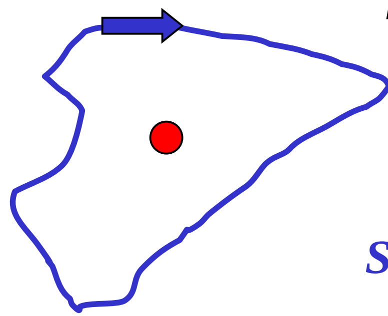
$$\vec{j} = -\frac{e^2 n_s}{mc}\vec{A}$$ →

Divergence in the supercurrent dependence vs coordinate

contradiction ?

We need to introduce singularities of the order parameter phase to avoid divergence of current density

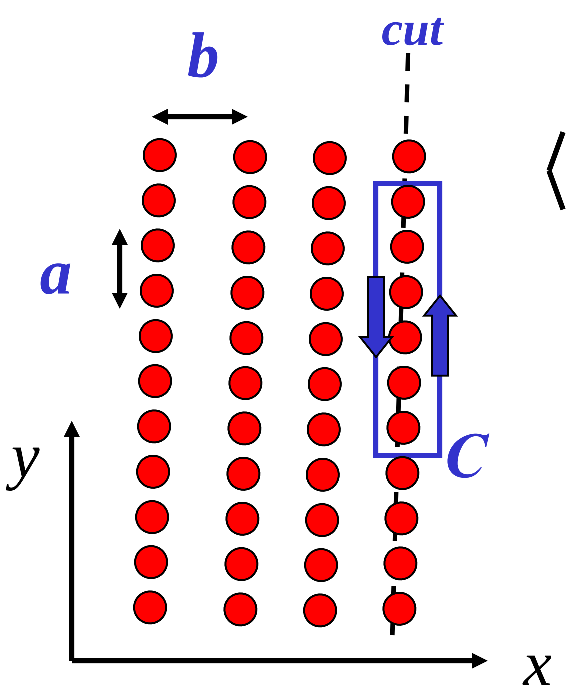
$$\vec{j}=\frac{e\hbar}{m}\left|\Psi\right|^{2}\left(-\frac{2e}{\hbar c}\vec{A}+\nabla\varphi\right)$$



$$rot\nabla\varphi\neq 0$$

$$\oint\nabla\varphi dl=2\pi n$$

Simplest version $n=1$



$$\left\langle\Delta\left(\nabla\varphi\right)\right\rangle_{y}=\frac{1}{Na}\oint_{C}\nabla\varphi dl=\frac{2\pi N}{Na}=\frac{2\pi}{a}$$

$$\vec{A}=\vec{y}_{0}\overline{B}_{z}x+\tilde{\vec{A}}$$

$$\frac{2e}{\hbar c}\overline{B}_{z}b=\frac{2\pi}{a}$$ **- compensation**

$$\overline{B}_{z}S=\int B_{z}ds=\frac{\pi\hbar c}{e}=\phi_{0}$$

Magnetic flux quantum

Reminder.

Gauge invariance

?

$$\nabla\Psi \to \nabla\Psi - \frac{2ie}{\hbar c}\vec{A}\Psi$$

$$\tilde{F} = F_n + \int\left(\frac{\hbar^2}{4m}\left|\left(\nabla - \frac{2ie}{\hbar c}\vec{A}\right)\Psi\right|^2 + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{\vec{B}^2}{8\pi} - \frac{\vec{B}\vec{H}}{4\pi}\right)dV$$

Energy of magnetic field

In the external field $\vec{H}$

Variation of the functional

Over Ψ^* → **Ginzburg – Landau equations**

Over $\vec{A}$ → **Maxwell equations**

London theory. Reminder

$$E_{kin} = n_s \frac{mv_s^2}{2} = \frac{mj_s^2}{2n_s e^2} = \frac{\lambda^2}{8\pi}\left(rot\vec{B}\right)^2 \qquad \lambda^2 = \frac{mc^2}{4\pi n_s e^2}$$

$$F_{sB} = F_{s0} + \frac{1}{8\pi}\int\left(\vec{B}^2 + \lambda^2\left(rot\vec{B}\right)^2\right)dV$$

$$rot\vec{B} = \frac{4\pi}{c}\left(-\frac{c}{4\pi\lambda^2}\vec{A}\right)$$

$$F_{sB} = F_{s0} + \frac{1}{8\pi}\int\left(\left(rot\vec{A}\right)^2 + \lambda^{-2}\vec{A}^2\right)dV$$

London theory. Reminder.

$$\delta F_{sB} = \frac{1}{4\pi}\int\left(\vec{B}\delta\vec{B} + \lambda^2 rot\vec{B} rot\,\delta\vec{B}\right)dV = \frac{1}{4\pi}\int\left(\vec{B}\delta\vec{B} + \lambda^2 rotrot\vec{B}\cdot\delta\vec{B}\right)dV$$

$$div\left[rot\vec{B},\delta\vec{B}\right] = -rot\,\delta\vec{B} rot\vec{B} + rotrot\vec{B}\cdot\delta\vec{B}$$

$$\vec{B} + \lambda^2 rotrot\vec{B} = 0$$

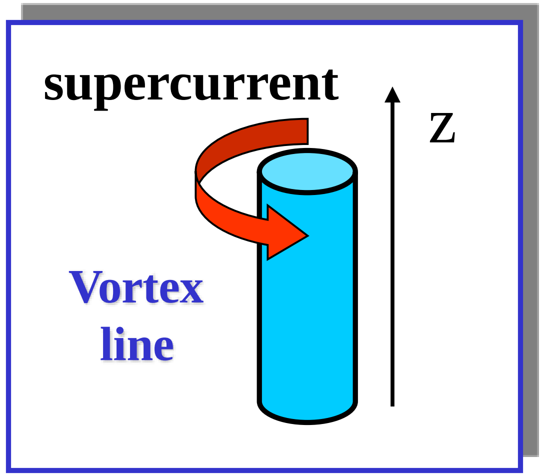
$$div\vec{A} = 0$$

$$\vec{A} + \lambda^2 rotrot\vec{A} = 0$$

$$\vec{n}\vec{A} = 0$$

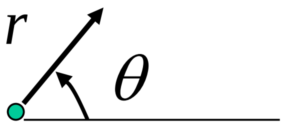
Isolated vortex. Low magnetic field

$$rot\nabla\varphi = 2\pi\delta(\vec{r})\vec{z}_0$$

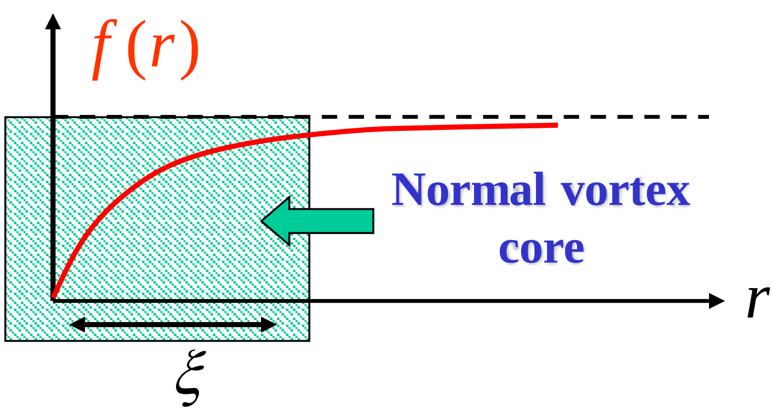


Order parameter structure

$$\Psi = f(r)e^{i\theta}$$



$f(0) = 0$ ⇐ **singlevalued** Ψ



Magnetic field distribution

$\kappa >> 1 \qquad r >> \xi$

$$-\lambda^2\Delta\vec{B} + \vec{B} = \phi_0\delta(\vec{r})\vec{z}_0$$

$$\vec{B} = \frac{\phi_0}{2\pi\lambda^2}K_0(r/\lambda)\vec{z}_0$$

MacDonald function

$r >> \lambda \qquad \vec{B} \approx \frac{\phi_0\vec{z}_0}{\sqrt{8\pi r\lambda^3}}e^{-r/\lambda}$

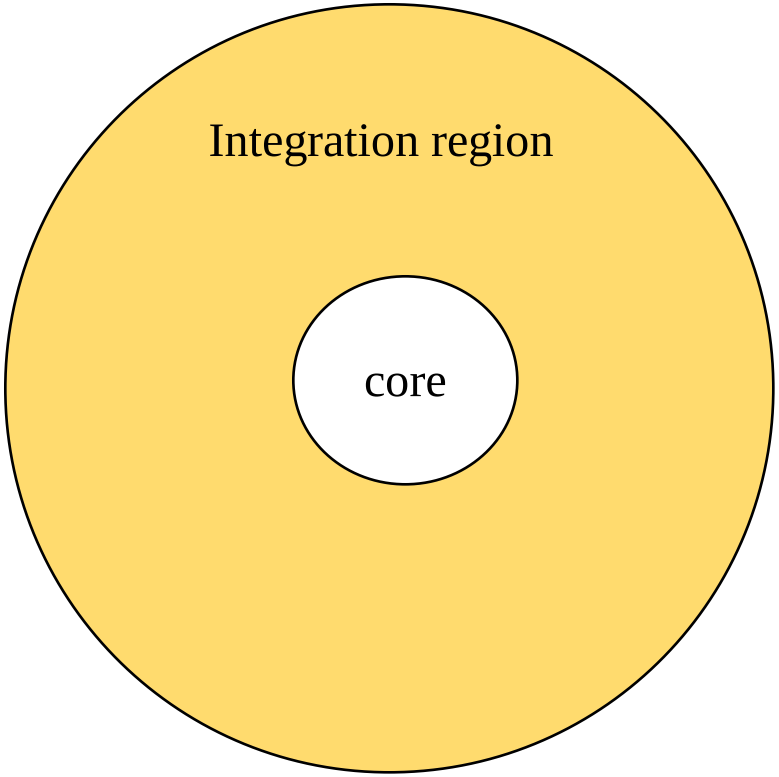
$\xi << r << \lambda \qquad \vec{B} \approx \frac{\phi_0\vec{z}_0}{2\pi\lambda^2}\ln\left(\frac{\lambda}{r}\right)$

Isolated vortex. Low magnetic fields. Vortex energy.

$$F_{sB} = F_{s0} + \frac{1}{8\pi}\int\left(\vec{B}^2 + \lambda^2\left(rot\vec{B}\right)^2\right)dV$$

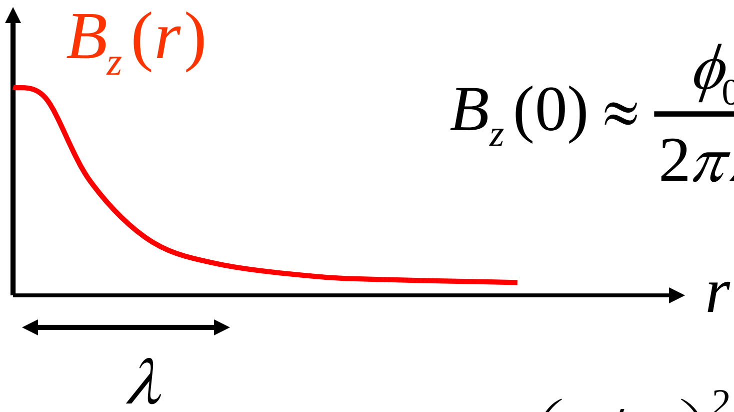
$$\varepsilon_v = \frac{1}{8\pi}\int \vec{B}\left(\vec{B} + \lambda^2 rotrot\vec{B}\right)dS + \frac{\lambda^2}{8\pi}\oint\left[\vec{B}, rot\vec{B}\right]ds$$

$$\varepsilon_v = \frac{1}{8\pi}\int \left|\vec{B}\right|\phi_0\delta(\vec{r})dS + \frac{\lambda^2}{8\pi}\oint\left[\vec{B}, rot\vec{B}\right]ds$$



$$\varepsilon_v = \frac{\lambda^2}{8\pi}\left(B \cdot 2\pi r\frac{dB}{dr}\right)\Bigg|_{\xi}$$

$$\varepsilon_v = \frac{\phi_0}{8\pi}B(0) = \left(\frac{\phi_0}{4\pi\lambda}\right)^2 \ln\kappa \sim \frac{H_{cm}^2}{8\pi}\xi^2$$



$$B_z(0) \approx \frac{\phi_0}{2\pi\lambda^2} \ln\left(\frac{\lambda}{\xi}\right)$$

Free energy of the vortex line per unit length

$$\varepsilon_v \approx \left(\frac{\phi_0}{4\pi\lambda}\right)^2 \left(\ln\left(\frac{\lambda}{\xi}\right) + \gamma\right)$$

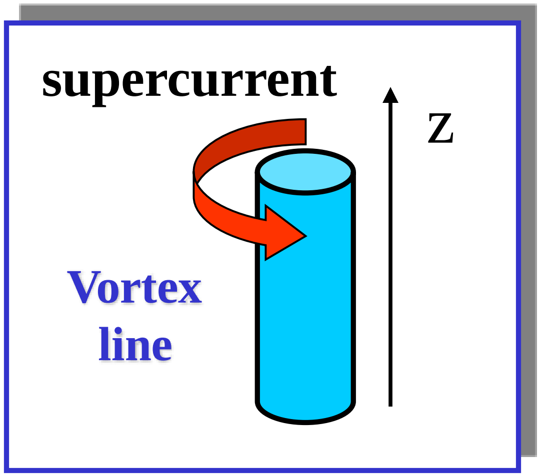
$$F = \varepsilon_v - \frac{1}{4\pi}\int \vec{B}\vec{H}\,dV = \varepsilon_v - \frac{H\phi_0}{4\pi}$$

Lower critical field $$H_{c1} = \frac{4\pi\varepsilon_v}{\phi_0}$$

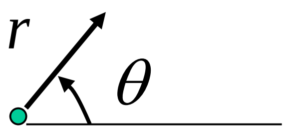
Vortices become energetically favorable for $H > H_{c1}$

(in the infinite cylinder in parallel field)

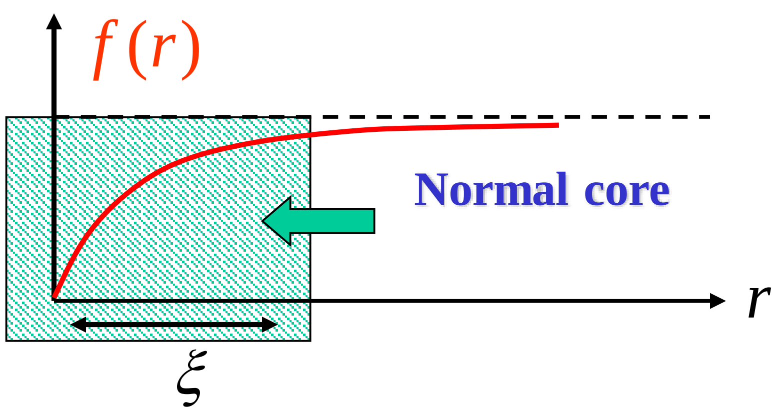
Isolated vortex. Core structure



$$\Psi = f(r)e^{i\theta}$$



$f(0) = 0$ ⇐ singlevalued Ψ



$$\mathbf{A} = A(r)\hat{\boldsymbol{\theta}},$$

$$A(r) = \frac{1}{r}\int_0^r r'h(r')\,dr'$$

Near the core

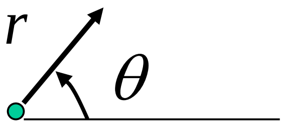
$$A(r) = (h(0)\,r)/2,$$

Far from the core

$$A_\infty = \Phi_0/2\pi r,$$

Isolated vortex. Core structure.

$$\Psi = f(r)e^{i\theta}$$

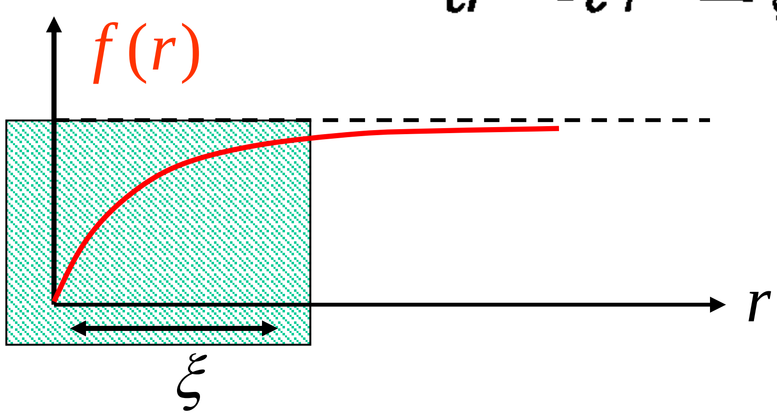


$$f - f^3 - \xi^2\left[\left(\frac{1}{r} - \frac{2\pi A}{\Phi_0}\right)^2 f - \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right] = 0.$$

$$f - f^3 - \xi^2\left[\left(\frac{1}{r} - \frac{\pi h(0)\, r}{\Phi_0}\right)^2 f - \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right] = 0$$

$$f \approx cr^n$$

$$cr^n - c^3r^{3n} - \xi^2\left[\left(\frac{1}{r} - \frac{\pi h(0)\, r}{\Phi_0}\right)^2 cr^n - n^2cr^{n-2}\right] = 0.$$



$$f \approx cr\left\{1 - \frac{r^2}{8\xi^2}\left[1 + \frac{h(0)}{H_{c2}}\right]\right\}$$

Interaction of vortices. Lorentz force

$$\mathbf{h}(\mathbf{r}) = \mathbf{h}_1(\mathbf{r}) + \mathbf{h}_2(r) = \hat{\mathbf{z}}\,[h(|\mathbf{r}-\mathbf{r}_1|) + h(|\mathbf{r}-\mathbf{r}_2|)]$$

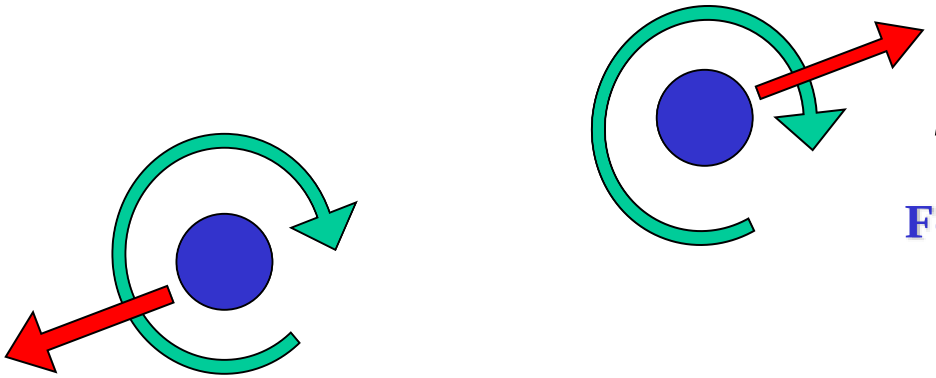
$$\Delta F = (\Phi_0/8\pi)\,[h_1(\mathbf{r}_1) + h_1(\mathbf{r}_2) + h_2(\mathbf{r}_1) + h_2(\mathbf{r}_2)] =$$
$$= 2\,(\Phi_0/8\pi)\,h_1(\mathbf{r}_1) + (\Phi_0/4\pi)\,h_1(\mathbf{r}_2).$$

$$F_{12} = \frac{\Phi_0 h_1(\mathbf{r}_2)}{4\pi} = \frac{\Phi_0^2}{8\pi^2\lambda^2}\,K_0\left(\frac{r_{12}}{\lambda}\right)$$

$$f_{2x} = -\frac{\partial F_{12}}{\partial x_2} = -\frac{\Phi_0}{4\pi}\,\frac{\partial h_1(\mathbf{r}_2)}{\partial x_2} = \frac{\Phi_0}{c}\,\mathbf{J}_{1y}(\mathbf{r}_2)$$

$$\mathbf{f} = (\mathbf{J}_s \times \Phi_0)/c$$

Interaction of vortices.



$$\vec{f} = \frac{\phi_0}{c}\left[\vec{j}, \vec{z}_0\right]$$

Force on a vortex in the field of transport current

$\vec{j}$

(Lorentz force)

Energy of vortex-vortex interaction

$$F_{12} = \frac{\phi_0^2}{8\pi^2\lambda^2} K_0\left(r_{12}/\lambda\right)$$

Vortex lattice.

$$G - G_{s0} = \frac{B}{\Phi_0}\varepsilon_1 + \sum_{i>j} F_{ij} - \frac{BH}{4\pi}$$

$$\frac{\partial G}{\partial B} = 0 = \frac{\varepsilon_1}{\Phi_0} - \frac{H}{4\pi} + \frac{\partial}{\partial B}\sum_{i>j} F_{ij}$$

$$\frac{\partial}{\partial B}\sum_{i>j} F_{ij} = \frac{1}{4\pi}(H - H_{c1})$$

$$\mathbf{a}_1 = a_\Delta \hat{\mathbf{x}}, \quad \mathbf{a}_2 = \frac{1}{2} a_\Delta (\hat{\mathbf{x}} + \sqrt{3}\,\hat{\mathbf{y}})$$

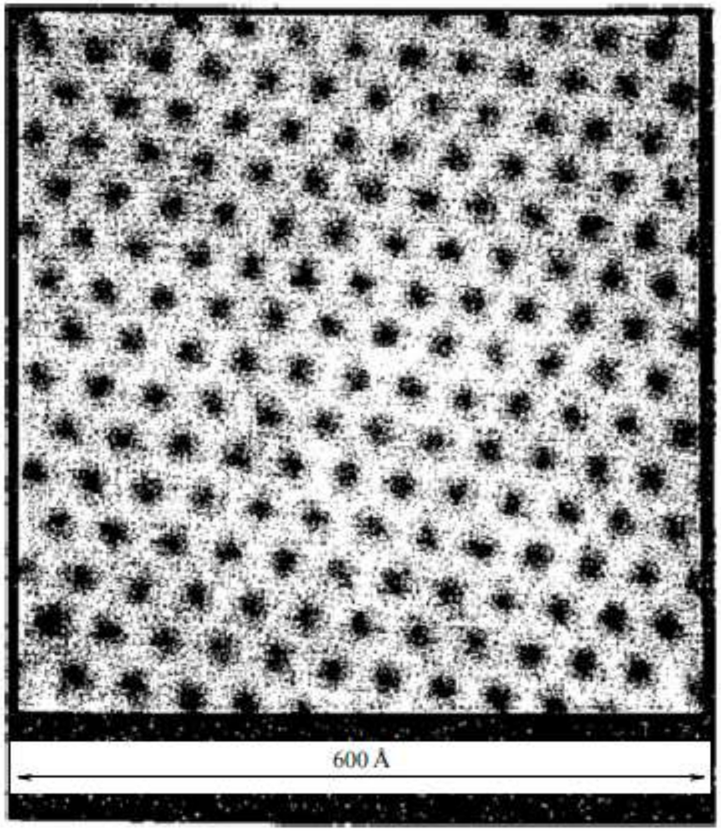
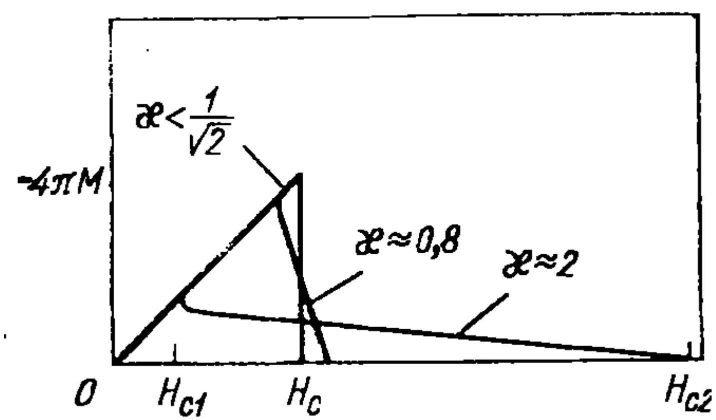


Рис. 6. Вихри в $NbSe_2$, наблюдаемые методом сканирующей туннельной микроскопии.



Strong magnetic fields. Upper critical field

$$-\frac{\hbar^2}{4m}\left(\nabla - \frac{2ie}{\hbar c}\vec{A}\right)^2 \Psi = -a\Psi$$

energy E

$$\vec{A} = -\vec{x}_0 H_z y$$

Min E → **Max critical T (H)**

$$E_0 = \frac{e\hbar H}{2mc} \Rightarrow H_{c2} = \frac{\phi_0}{2\pi\xi^2}$$

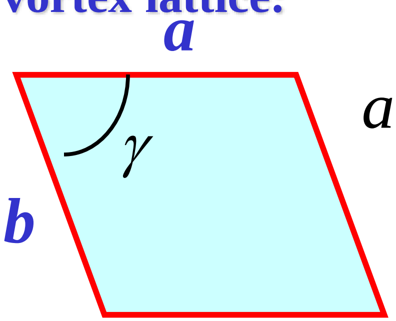
Superconducting nucleous

$$\Psi = A\exp\left(ikx - \frac{(y-kL_H^2)^2}{2L_H^2}\right)$$

Magnetic length $L_H = \sqrt{\frac{\hbar c}{2eH}}$

Vortex lattice. Abrikosov problem (1957)

Primitive cell of the vortex lattice:



$$ab\sin\gamma = 2\pi L_H^2$$

$$\rho = \frac{b\cos\gamma}{a} \qquad \sigma = \frac{b\sin\gamma}{a}$$

$$\Psi = A\sum_n \exp\left(i\pi\rho n(n-1) + \frac{i2\pi nx}{a} - \frac{(y-n\sigma a)^2}{2L_H^2}\right)$$

Vortex lattice. Abrikosov problem(1957)

$$\Psi = C\sum_n \exp\left(i\pi\rho n(n-1) + \frac{i2\pi nx}{a} - \frac{(y-n\sigma a)^2}{2L_H^2} \right) = C\psi$$

$$\tilde{F} = F_n + \int\left(\Psi^*\left(-\frac{\hbar^2}{4m}\left(\nabla - \frac{2ie}{\hbar c}\vec{A}\right)^2 \Psi + a\Psi \right) + \frac{b}{2}|\Psi|^4 \right) dV$$

$$\lambda >> \xi$$

$$-\frac{\hbar^2}{4m}\left(\nabla - \frac{2ie}{\hbar c}\vec{A}\right)^2 \Psi = \frac{\hbar eH}{2mc}\Psi$$

$$\tilde{F} - F_n = V|a|\left(C^2\left\langle|\psi|^2\right\rangle\left(\frac{\hbar eH}{2mc|a|} - 1\right) + \frac{b}{2|a|}C^4\left\langle|\psi|^4\right\rangle \right) = V|a|\left(C^2\left\langle|\psi|^2\right\rangle\left(\frac{H}{H_{c2}} - 1\right) + \frac{b}{2|a|}C^4\left\langle|\psi|^4\right\rangle \right)$$

$$C\left\langle|\psi|^2\right\rangle\left(\frac{H}{H_{c2}} - 1\right) + \frac{b}{|a|}C^3\left\langle|\psi|^4\right\rangle = 0$$

$$C^2 = -\frac{|a|}{b}\frac{\left\langle|\psi|^2\right\rangle}{\left\langle|\psi|^4\right\rangle}\left(\frac{H}{H_{c2}} - 1\right)$$

$$\tilde{F} - F_n = V\frac{|a|^2}{b}\left(-\frac{\left\langle|\psi|^2\right\rangle^2}{\left\langle|\psi|^4\right\rangle}\left(\frac{H}{H_{c2}} - 1\right)^2 + \frac{1}{2}C^4\frac{\left\langle|\psi|^2\right\rangle^2}{\left\langle|\psi|^4\right\rangle}\left(\frac{H}{H_{c2}} - 1\right)^2 \right)$$

Vortex latice. Abrikosov problem(1957)

$$\Psi = C\sum_n \exp\left(i\pi\rho n(n-1) + \frac{i2\pi nx}{a} - \frac{(y-n\sigma a)^2}{2L_H^2} \right) = C\psi$$

$$\tilde{F} - F_n = -V\frac{|a|^2}{2b}\frac{\left\langle |\psi|^2 \right\rangle^2}{\left\langle |\psi|^4 \right\rangle}\left(\frac{H}{H_{c2}} - 1\right)^2$$

Square lattice:

$\rho = 0 \qquad \sigma = 1$

Saddle point

$$\min_{\rho,\sigma}\frac{\left\langle |\Psi|^4 \right\rangle}{\left\langle |\Psi|^2 \right\rangle^2} = ?$$

Hexagonal lattice:

$\rho = \frac{1}{2} \qquad \sigma = \frac{\sqrt{3}}{2}$

Minimum of energy

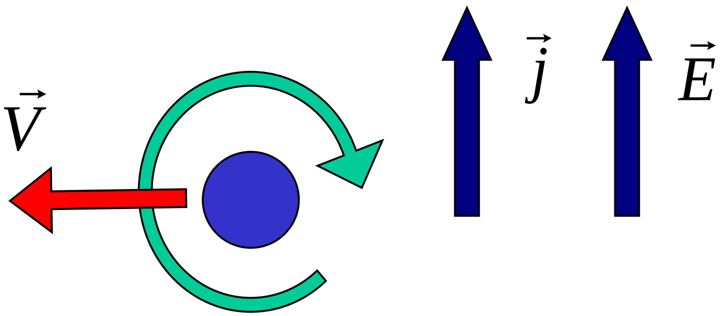
Tranport properties of vortex state. Vortex matter.

Viscous flux flow.

$$\eta\vec{V} = \frac{\phi_0}{c}\left[\vec{j},\vec{z}_0\right] + \vec{F}$$

viscosity

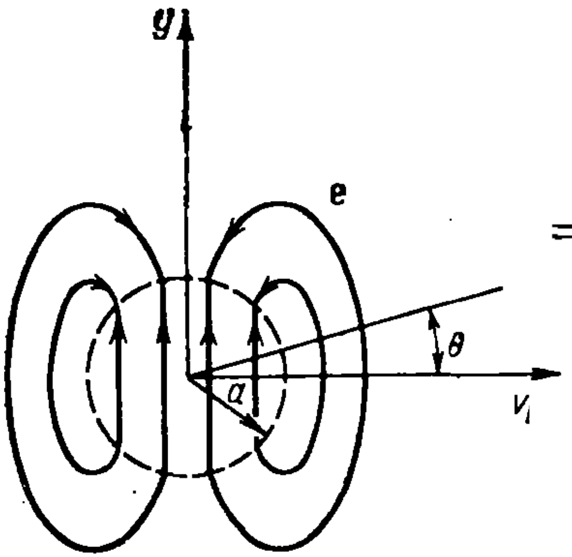
$$\left\langle \vec{E} \right\rangle = \frac{1}{c}\left[\vec{V},\vec{B}\right]$$



$$\left\langle \vec{E} \right\rangle = \frac{\phi_0 B}{\eta c^2}\vec{j}$$

$$\rho = \rho_n \frac{H}{H_{c2}}$$

Transport properties of the vortex state. Bardeen – Stephen problem



$$\mathbf{e} = \frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = \frac{\partial}{\partial t}\left(\frac{m^* \mathbf{v}_s}{e^*}\right) = -\mathbf{v}_L \cdot \nabla\left(\frac{m^* \mathbf{v}_s}{e^*}\right) = -\mathbf{v}_L \cdot \nabla\left(\frac{\hbar \hat{\theta}}{2er}\right).$$

$$\mathbf{e} = -\left(\frac{v_{Lx}\Phi_0}{2\pi c}\right)\frac{\partial}{\partial x}\left(\frac{\hat{\theta}}{r}\right) = \left(\frac{v_{Lx}\Phi_0}{2\pi c r^2}\right)(\cos\theta\,\hat{\theta} - \sin\theta\,\hat{\mathbf{r}}).$$

$$\mathbf{e}_{\text{внутр}} = \frac{v_{Lx}\Phi_0}{2\pi a^2 c}\hat{\mathbf{y}}.$$

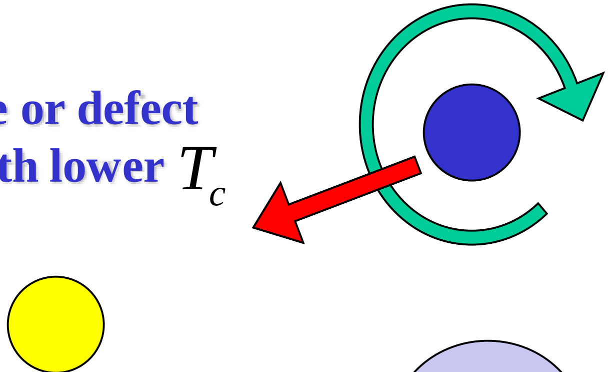
$$W_{\text{внутр}} = \pi a^2 \sigma_n e^2_{\text{внутр}} = \frac{v_L^2 \Phi_0^2}{4\pi a^2 c^2 \rho_n}$$

$$\eta = \frac{\Phi_0^2}{2\pi a^2 c^2 \rho_n} \approx \frac{\Phi_0 H_{c2}}{\rho_n c^2}$$

$$\frac{\rho_f}{\rho_n} = \frac{2\pi a^2 B}{\Phi_0} = \left(\frac{a}{\xi}\right)^2 \frac{B}{H_{c2}} \approx \frac{B}{H_{c2}}$$

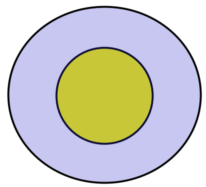
Vortex pinning

Hole or defect with lower T_c

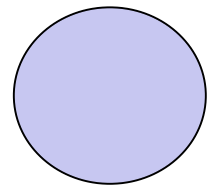


$$\eta \vec{V} = \frac{\phi_0}{c}\left[\vec{j}, \vec{z}_0\right] + \vec{F}$$

$$\vec{F} = -\nabla U(\vec{r})$$



$$E_{core} \sim \left(\xi^2 - R^2\right)\frac{H_{cm}^2}{8\pi}$$

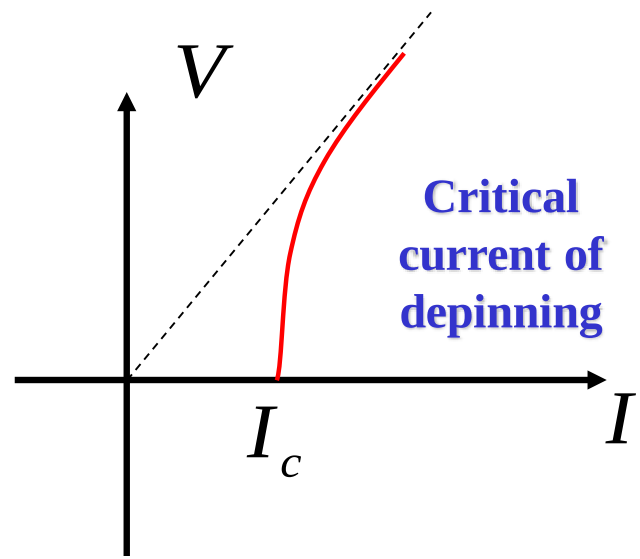


$$E_{core} \sim \xi^2 \frac{H_{cm}^2}{8\pi}$$

Electrostatic analogy

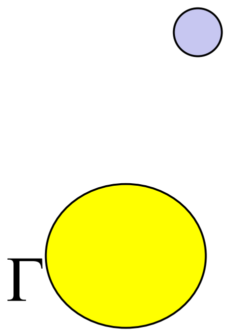
$$\vec{j} \sim \nabla \varphi \qquad \Delta \varphi = 0$$

$$div\vec{j} = 0 \qquad rot\nabla\varphi = 2\pi\delta(\vec{r} - \vec{r}_1)$$



Critical current of depinning

Electrostatic analogy



$$\left.\frac{\partial\varphi}{\partial\vec{n}}\right|_{\Gamma} = 0$$

$$\vec{j} \sim \nabla\varphi$$

$$div\,\vec{j} = 0$$

$$R >> \xi$$

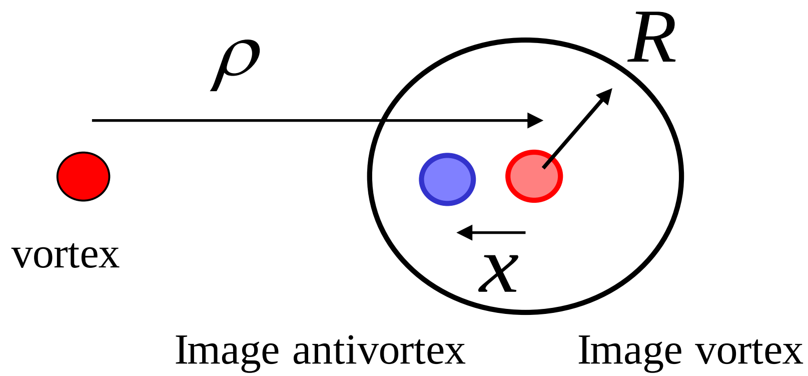
$$\nabla\varphi = \left[\vec{z}_0, \nabla\chi\right]$$

$$\Delta\varphi = 0$$

$$rot\nabla\chi = 0$$

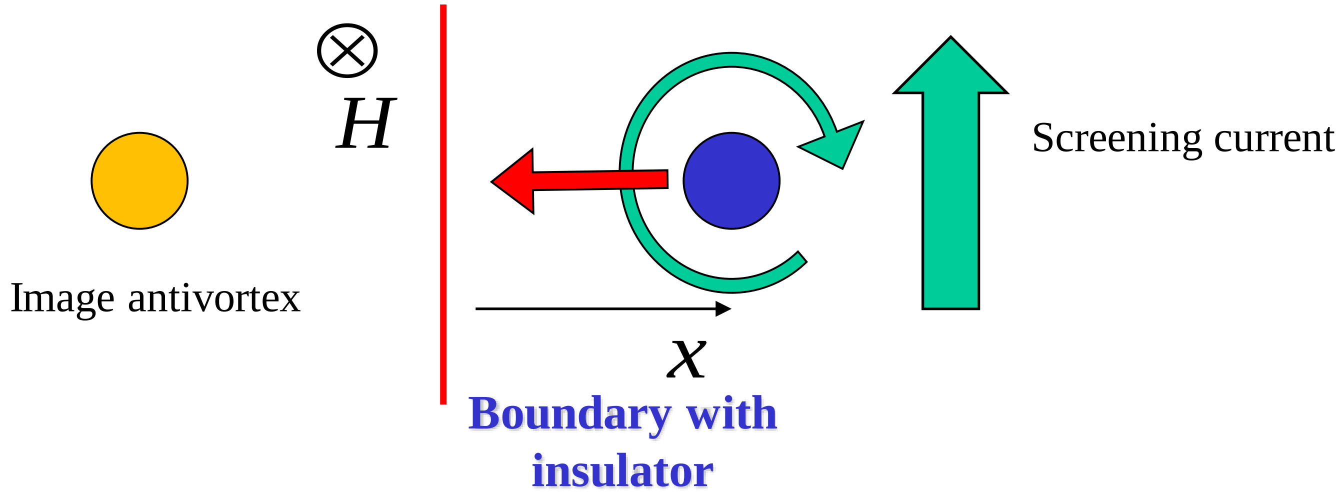
$$rot\nabla\varphi = 2\pi\delta(\vec{r} - \vec{r}_1)$$

$$div\nabla\chi = 2\pi\delta(\vec{r} - \vec{r}_1)$$



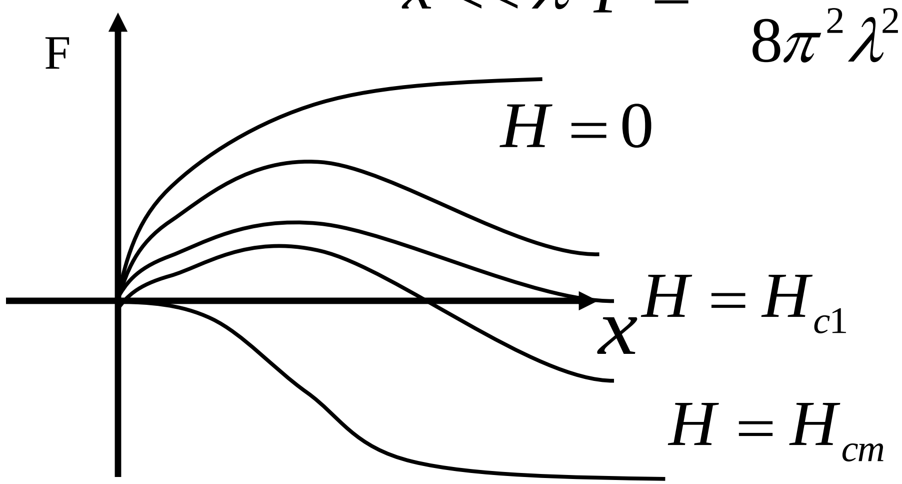
$$x\rho = R^2$$

Vortex pinning. Bean – Livingstone barrier



$$F = -\frac{\phi_0^2}{8\pi^2\lambda^2} K_0\left(2x/\lambda\right) + \frac{\phi_0}{4\pi} H_0 e^{-x/\lambda} + \frac{\phi_0}{4\pi}\left(H_{c1} - H_0\right)$$

$$x \ll \lambda \quad F = -\frac{\phi_0^2}{8\pi^2\lambda^2} \ln\frac{\lambda}{2x} + \frac{\phi_0}{4\pi} H_0 + \frac{\phi_0}{4\pi}\left(H_{c1} - H_0\right)$$

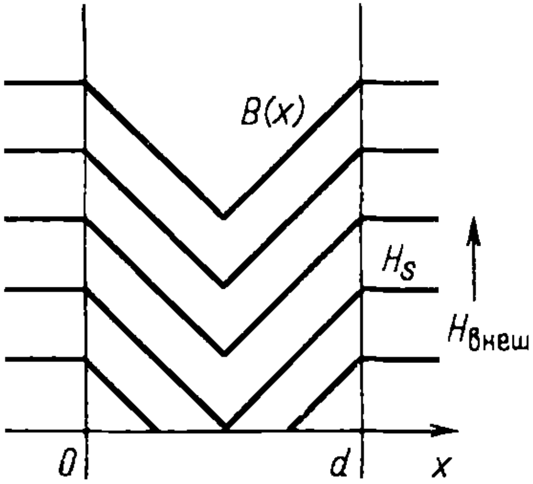


Critical state. Bean model

$$\eta\vec{V} = \frac{\phi_0}{c}\left[\vec{j}, \vec{z}_0\right] + \vec{F}$$

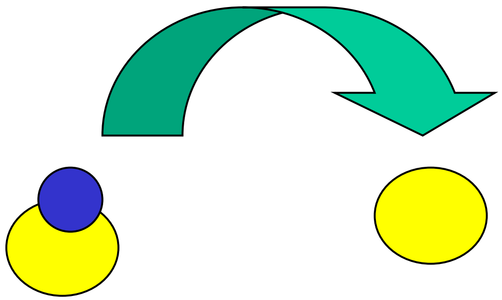
$$\vec{V} = 0$$

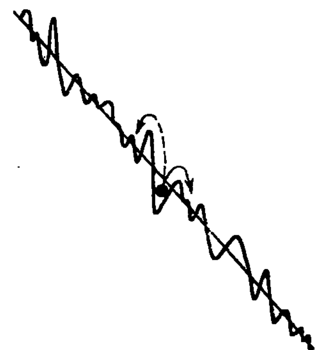
$$\frac{\phi_0}{c}\left[\vec{j}, \vec{z}_0\right] + \vec{F} = 0$$

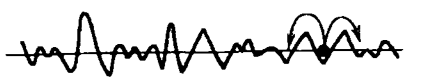


$$\vec{j} = \frac{c}{4\pi} rot\vec{B}$$

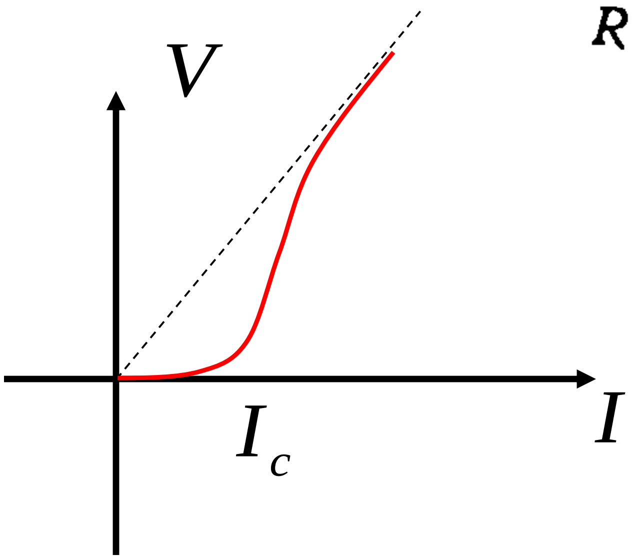
Vortex creep.



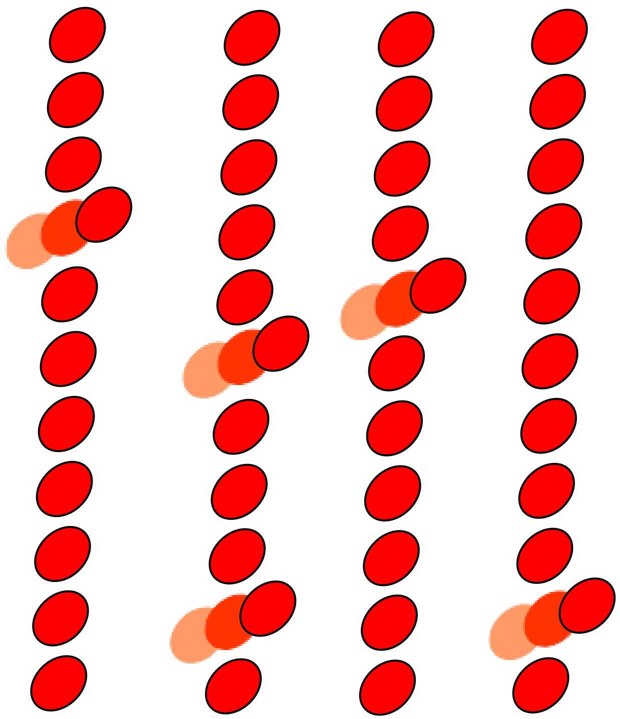




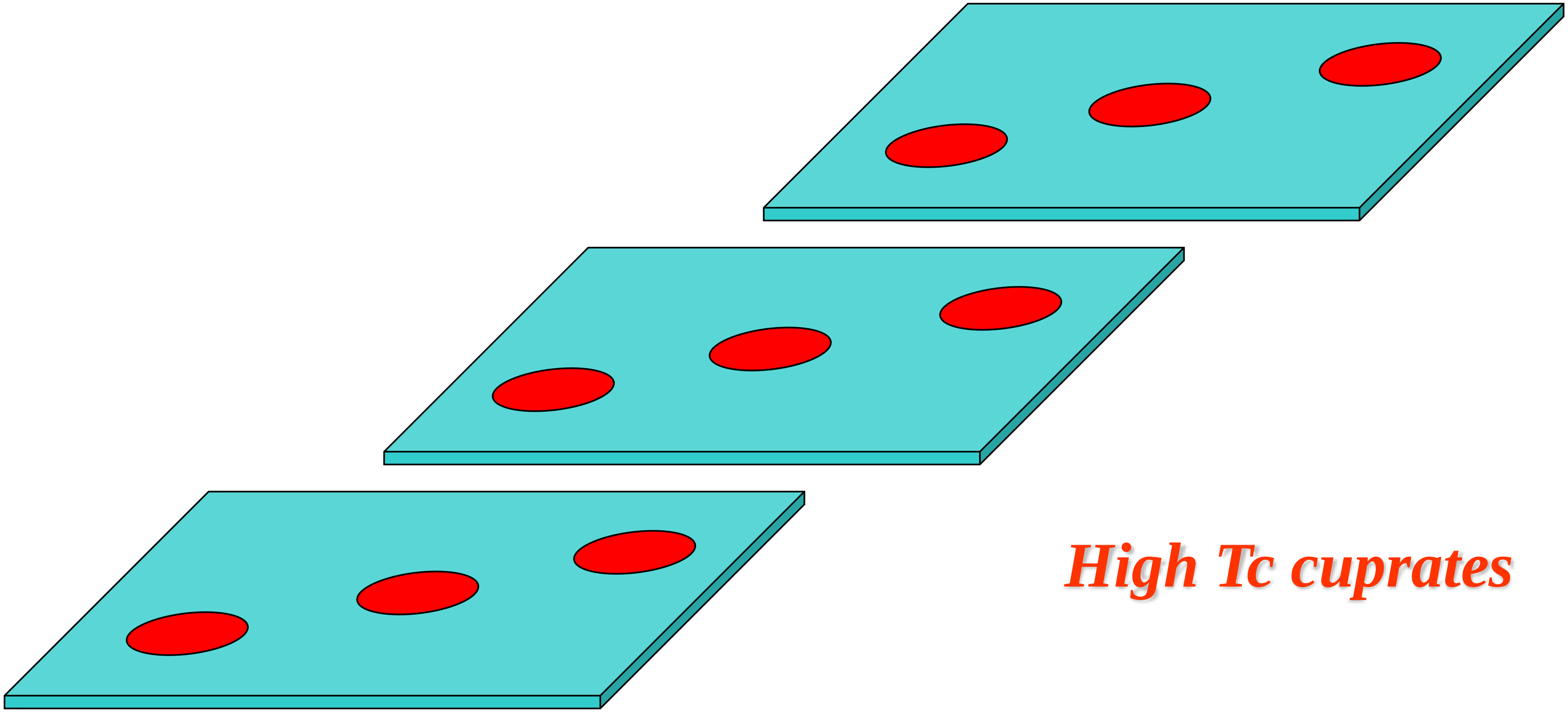
$$R = \omega_0 \exp\{-F_0/kT\}$$



Vortex lattice melting.

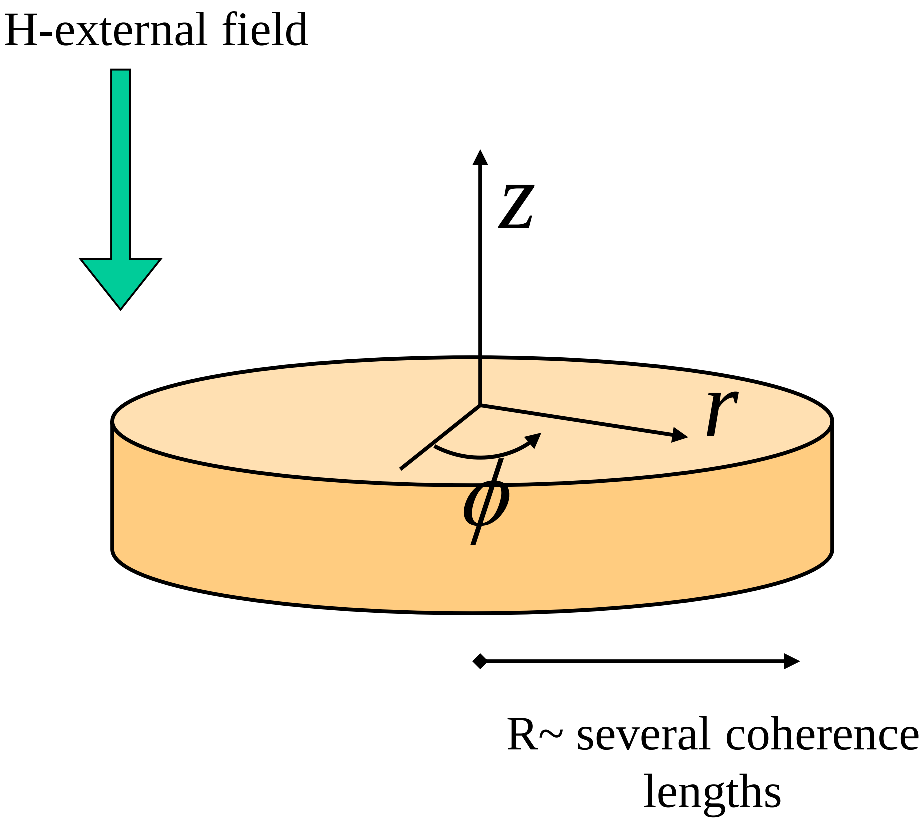


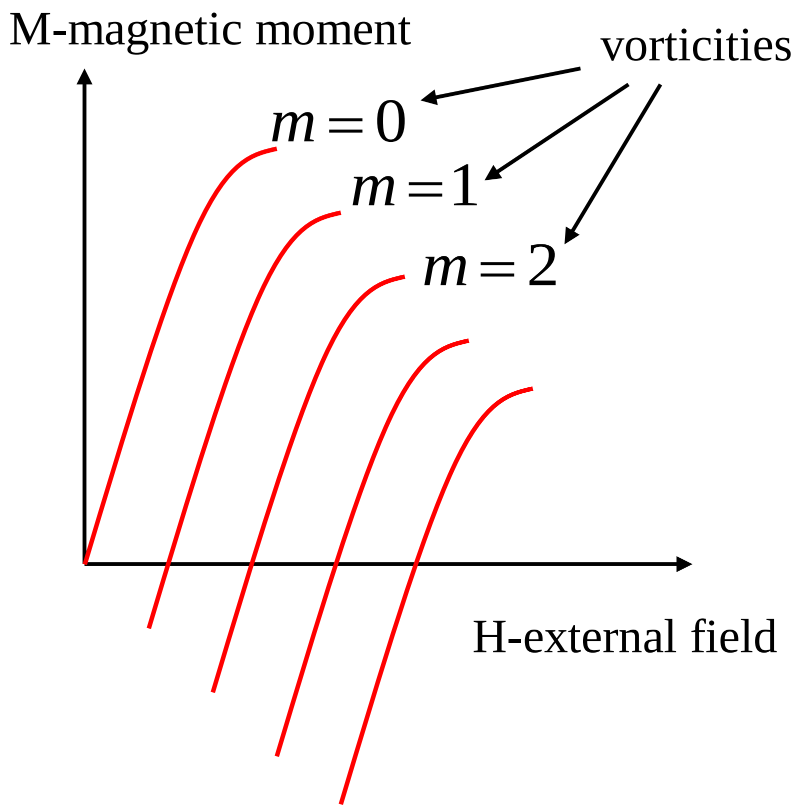
Layered superconductors



High Tc cuprates

Vortices in mesoscopic samples





problems

1. Find lower critical field in type-II superconductor

2. Find upper critical field in type-II superconductor

3. Find the surface barrier for the vortex entry

4. Find lower critical field in type-II superconductor with anisotropic mass tensor

5. Find upper critical field in type-II superconductor with anisotropic mass tensor

6. Find the vortex viscosity

7. Find the distribution of magentic field for a vortex in a thin film (trapped perpendicular to the film plane)

8. Show that the energy of hexagonal vortex lattice gives us the minimum of free energy.

9. Find the energy of SN boundary

10. Find the upper critical field in a thin film for different field orientations