Microscopic theory of mesoscopic superconductivity

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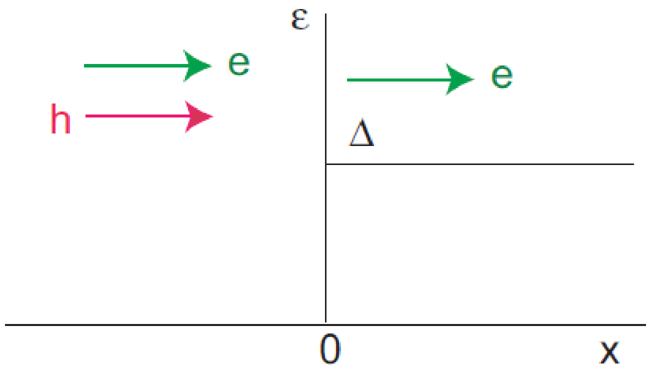
Outline of the 2nd section – Andreev reflection

- Andreev equations
- Andreev reflection
- Andreev states
- Supercurrent through a SNS junction
- Vortex core states

Andreev equations



$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} U(x) \\ V(x) \end{pmatrix}$$

where $|\mathbf{k}| = k_F$ while $U(x)$ and $V(x)$ vary slowly over distances of the order of k_F^{-1}. Inserting this into the BdG equations

$$-\frac{\hbar^2}{2m}\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2 u - \frac{\hbar^2 k_F^2}{2m} u + \Delta v = \epsilon u$$

$$\frac{\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar c}\mathbf{A}\right)^2 v + \frac{\hbar^2 k_F^2}{2m} v + \Delta^* u = \epsilon v$$

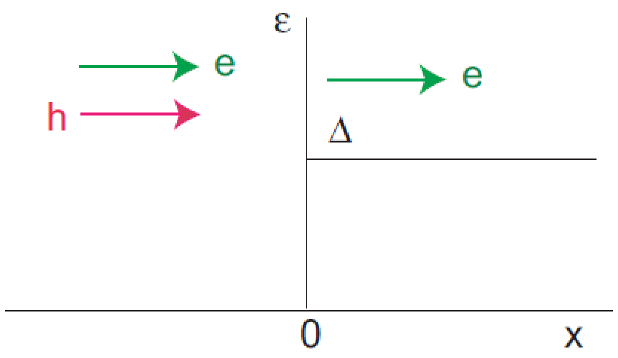
and neglecting the second derivatives of U and V, we find

$$-i\hbar \mathbf{v}_F \cdot \left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right) U + \Delta V = \epsilon U$$

$$i\hbar \mathbf{v}_F \cdot \left(\nabla + \frac{ie}{\hbar c}\mathbf{A}\right) V + \Delta^* U = \epsilon V$$

Andreev equations

Andreev reflection



$$-i\hbar v_x \frac{dU}{dx} + \Delta V = \epsilon U$$

$$i\hbar v_x \frac{dV}{dx} + \Delta^* U = \epsilon V$$

Consider $\epsilon > |\Delta|$

The reflection process $\boldsymbol{k} \to -\boldsymbol{k}$ is prohibited

In the normal region:

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix}_L = e^{i\lambda_N x} \begin{pmatrix} U_+^+(N) \\ V_+^+(N) \end{pmatrix} + a e^{-i\lambda_N x} \begin{pmatrix} U_+^-(N) \\ V_+^-(N) \end{pmatrix}$$

$$\lambda_N = \frac{\epsilon}{\hbar v_x}$$

$$U_\pm^+(N) = 1,\ V_\pm^+(N) = 0,\ U_\pm^-(N) = 0,\ V_\pm^-(N) = 1$$

From the continuity of the wave functions at the interface:

$$a = V/U\ ,\ c = 1/U$$

In the superconducting region:

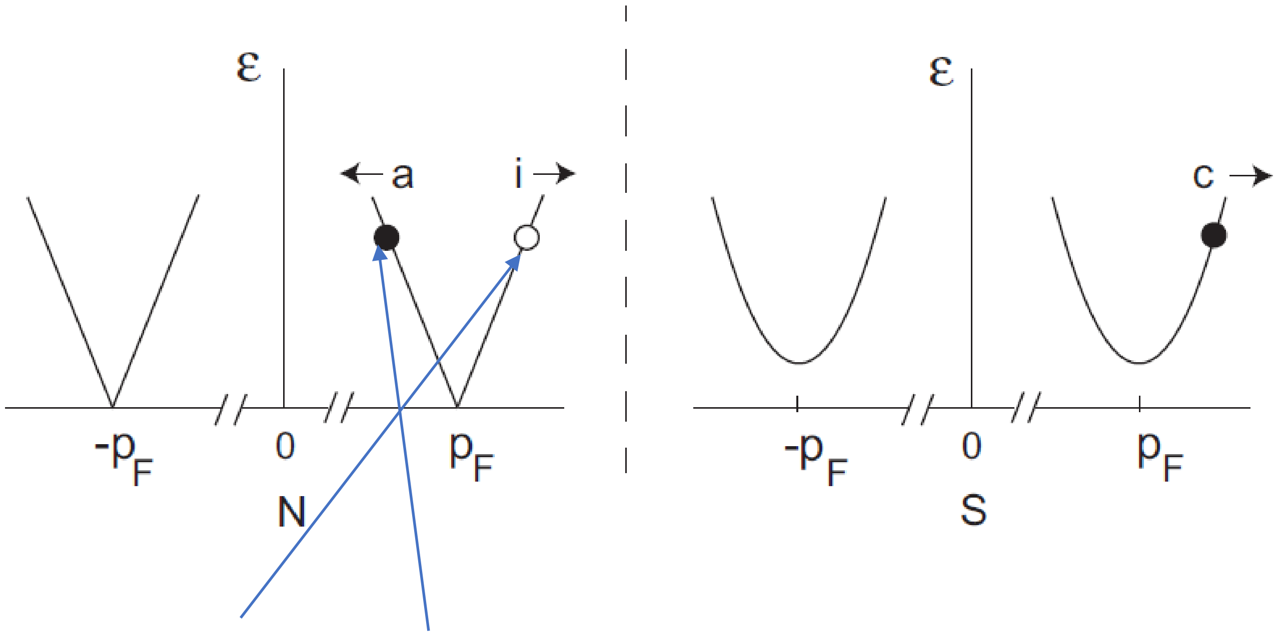
$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix}_R = c e^{i\lambda_S x} \begin{pmatrix} U_+^+(S) \\ V_+^+(S) \end{pmatrix}$$

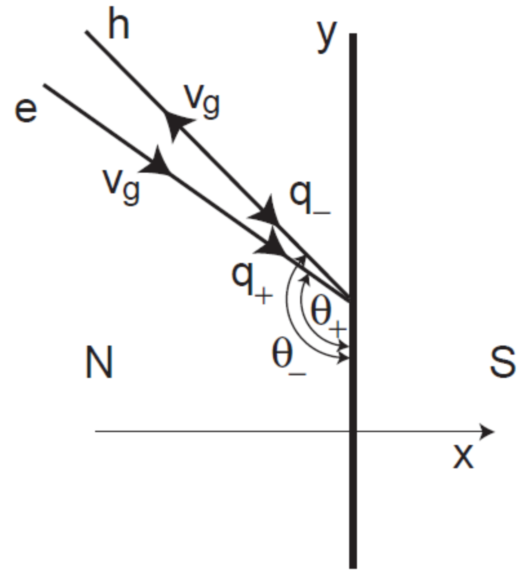
$$\lambda_S = \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\hbar v_x}$$

$$U_+^+(S) \equiv U,\ V_+^+(S) \equiv V$$

$$U^2 - V^2 = \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\epsilon}\ ,\ UV = \frac{|\Delta|}{2\epsilon}$$

Andreev reflection





$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{i\mathbf{q}_+\cdot\mathbf{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a e^{i\mathbf{q}_-\cdot\mathbf{r}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{q}_\pm = (k \pm \epsilon/\hbar v_F)\hat{\mathbf{k}}$$

For incident particle

$$q = k_F + \epsilon/\hbar v_F$$

$$v_g = v_F$$

For reflected hole

$$q = k_F - \epsilon/\hbar v_F$$

$$v_g = -v_F$$

$$|a|^2 + (U^2 - V^2)|c|^2 = 1$$

$$v_g(N)|a|^2 + v_g(S)|c|^2 = v_g(N)$$

$$v_g(S) = v_F(U^2 - V^2) = v_F \frac{\sqrt{\epsilon^2 - |\Delta|^2}}{\epsilon}$$

Andreev reflection

For the sub-gap energy $\epsilon < |\Delta|$, there are no states below the gap in the S region, thus the wave should decay for $x > 0$. The wave function on the right is

$$\left(\begin{array}{c} U(x) \\ V(x) \end{array}\right)_R = c e^{i\tilde{\lambda}_S x} \left(\begin{array}{c} \tilde{U}_+^+(S) \\ \tilde{V}_+^+(S) \end{array}\right)$$

where

$$\tilde{\lambda}_S = i \frac{\sqrt{|\Delta|^2 - \epsilon^2}}{\hbar v_x}$$

and

$$\tilde{U} = \frac{1}{\sqrt{2}} \left(\frac{\epsilon + i\sqrt{|\Delta|^2 - \epsilon^2}}{|\Delta|}\right)^{\frac{1}{2}}, \quad \tilde{V} = \frac{1}{\sqrt{2}} \left(\frac{\epsilon - i\sqrt{|\Delta|^2 - \epsilon^2}}{|\Delta|}\right)^{\frac{1}{2}}$$

The coefficients are

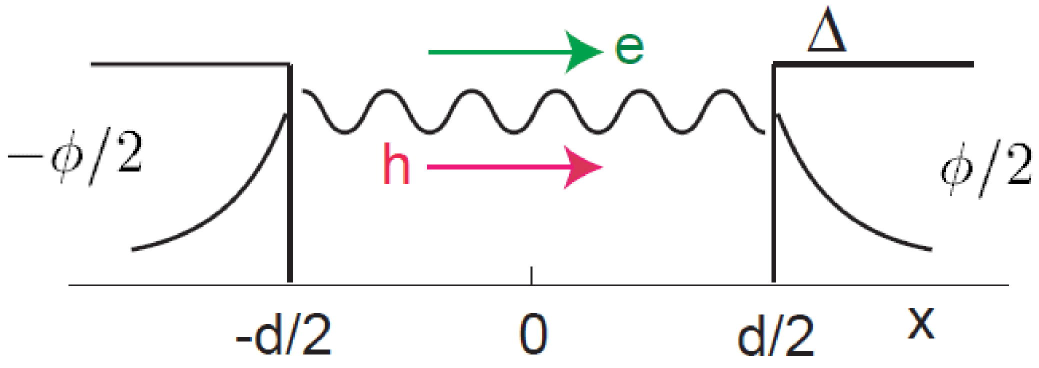
$$a = \tilde{V}/\tilde{U}\,, \quad c = 1/\tilde{U}$$

However, now

$$|a|^2 = 1$$

The Andreev reflection is complete since there are no transmitted particles.

Andreev states: SNS structures



Consider energy $\epsilon < |\Delta|$. A particle that moves in the normal region to the right will be Andreev reflected from the NS interface into a hole. The hole will then move to the left and is Andreev reflected into the particle, and so on. It is thus localized in the N region having a discrete energy spectrum.

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{i\mathbf{k}\cdot\mathbf{r}} \begin{pmatrix} U(x) \\ V(x) \end{pmatrix}$$

Andreev states: SNS structures

The wave function in the N region is $\begin{pmatrix} U(x) \\ V(x) \end{pmatrix}_N = A\left[e^{i\lambda_N x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + ae^{-i\lambda_N x}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$ $\qquad \lambda_N = \frac{\epsilon}{\hbar v_x}$

$k_x > 0$

in the right superconductor $x > d/2$ $\quad \begin{pmatrix} U(x) \\ V(x) \end{pmatrix}_R = d_1 e^{-\tilde{\lambda}_S x}\begin{pmatrix} \tilde{U}e^{i\phi/4} \\ \tilde{V}e^{-i\phi/4} \end{pmatrix}$ $\qquad \tilde{\lambda}_S = \frac{\sqrt{|\Delta|^2 - \epsilon^2}}{\hbar |v_x|}$

in the left superconductor $x < -d/2$ $\quad \begin{pmatrix} U(x) \\ V(x) \end{pmatrix}_L = d_1' e^{\tilde{\lambda}_S x}\begin{pmatrix} \tilde{V}e^{-i\phi/4} \\ \tilde{U}e^{i\phi/4} \end{pmatrix}$

Continuity at the right interface $\quad ae^{-i\lambda_N d} = \frac{\tilde{V}}{\tilde{U}}e^{-i\phi/2}$

Continuity at the left interface $\quad ae^{i\lambda_N d} = \frac{\tilde{U}}{\tilde{V}}e^{i\phi/2}$

$\Rightarrow \quad e^{2i(\lambda_N d - \phi/2)} = \frac{\epsilon + i\sqrt{|\Delta|^2 - \epsilon^2}}{\epsilon - i\sqrt{|\Delta|^2 - \epsilon^2}}$

denote $\quad \sin\alpha = \frac{\epsilon}{|\Delta|}$ $\quad$ then $\quad e^{2i(\lambda_N d - \phi/2)} = e^{-2i\alpha + i\pi}$ $\quad \Rightarrow \quad \boxed{\epsilon = \hbar\omega_x\left[\frac{\phi}{2} - \arcsin\frac{\epsilon}{|\Delta|} + \pi\left(l + \frac{1}{2}\right)\right]}$

$-\pi/2 < \alpha < \pi/2$ $\qquad \omega_x = \frac{v_x}{d} = t_x^{-1}$

Andreev states: SNS structures

$$k_x < 0 \qquad \epsilon = -\hbar|\omega_x|\left[\frac{\phi}{2} + \arcsin\frac{\epsilon}{|\Delta|} + \pi\left(l - \frac{1}{2}\right)\right]$$

Combining the both trajectories we obtain:

$$\epsilon = \pm\hbar|\omega_x|\left[\frac{\phi}{2} \mp \arcsin\frac{\epsilon}{|\Delta|} + \pi\left(l \pm \frac{1}{2}\right)\right]$$

Normalization of the wave function $|A|^2$ is determined from

$$\int \left(|u|^2 + |v|^2\right) dx = 1$$

In the normal region $|u|^2 + |v|^2 = 2|A|^2$. In the right region

$$|u|^2 + |v|^2 = |d_1|^2 e^{-2\lambda_S x}\left(|\tilde{U}|^2 + |\tilde{V}|^2\right) = 2|d_1|^2 e^{-2\lambda_S x}|\tilde{U}|^2$$

since $|\tilde{U}|^2 = |\tilde{V}|^2$. Employing the continuity of the wave functions $|A|^2 = |d_1|^2 e^{-\lambda_S d}|\tilde{U}|^2$ we find that in the right region

$$|u|^2 + |v|^2 = 2|A|^2 e^{-2\lambda_S(x-d/2)}$$

In the left region we similarly obtain

$$|u|^2 + |v|^2 = 2|A|^2 e^{-2\lambda_S(x+d/2)}$$

Calculating the integrals from $-\infty$ to $-d/2$ then from $-d/2$ to $d/2$ and from $d/2$ to ∞ we find

$$|A|^2 = \frac{1}{2(d + \lambda_S^{-1})} = \frac{1}{2}\frac{\sqrt{|\Delta|^2 - \epsilon^2}}{\hbar|v_x| + d\sqrt{|\Delta|^2 - \epsilon^2}}$$

Andreev states: SNS structures

$$\epsilon = \pm\hbar|\omega_x|\left[\frac{\phi}{2} \mp \arcsin\frac{\epsilon}{|\Delta|} + \pi\left(l \pm \frac{1}{2}\right)\right]$$

Consider the short junction limit: $d \ll \hbar|v_x|/|\Delta|$ ⟹ $\omega_x \gg |\Delta|$

For $v_x > 0$ we have to choose $l = -1$

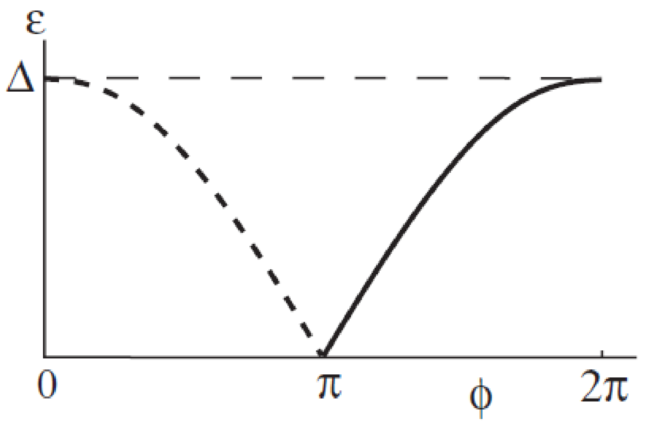
For $v_x < 0$, the choice is $l = 0$

$$\epsilon = \mp|\Delta|\cos\frac{\phi}{2}$$

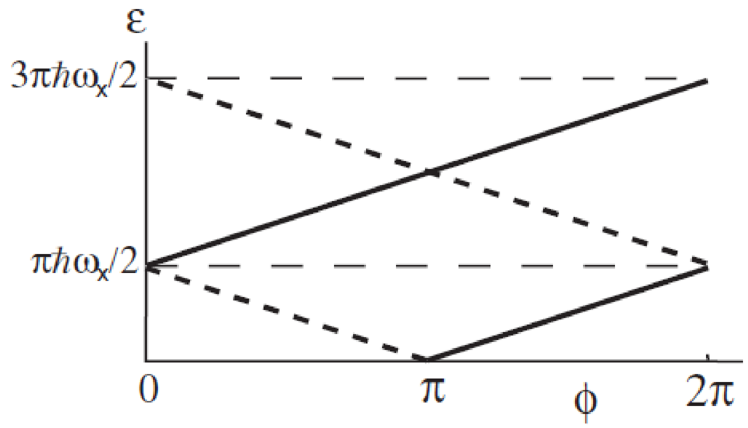
For the long junction limit: $d \gg \hbar|v_x|/|\Delta|$, $\omega_x \ll |\Delta|$

$$\epsilon = \pm\hbar|\omega_x|\left(\frac{\phi}{2} - \frac{\pi}{2}\right) + \pi\hbar|\omega_x|l$$

Short junction



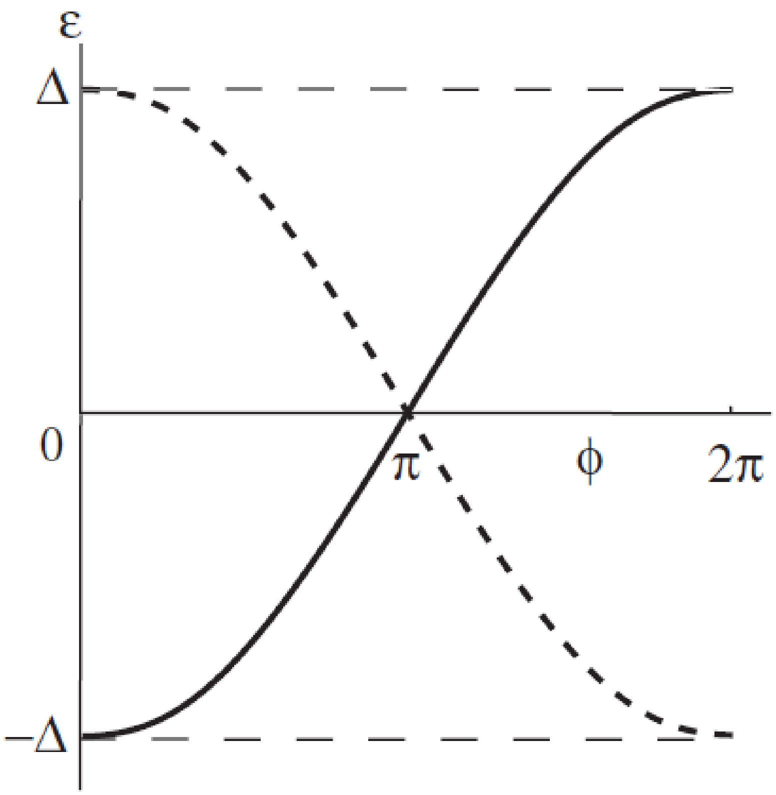
Long junction



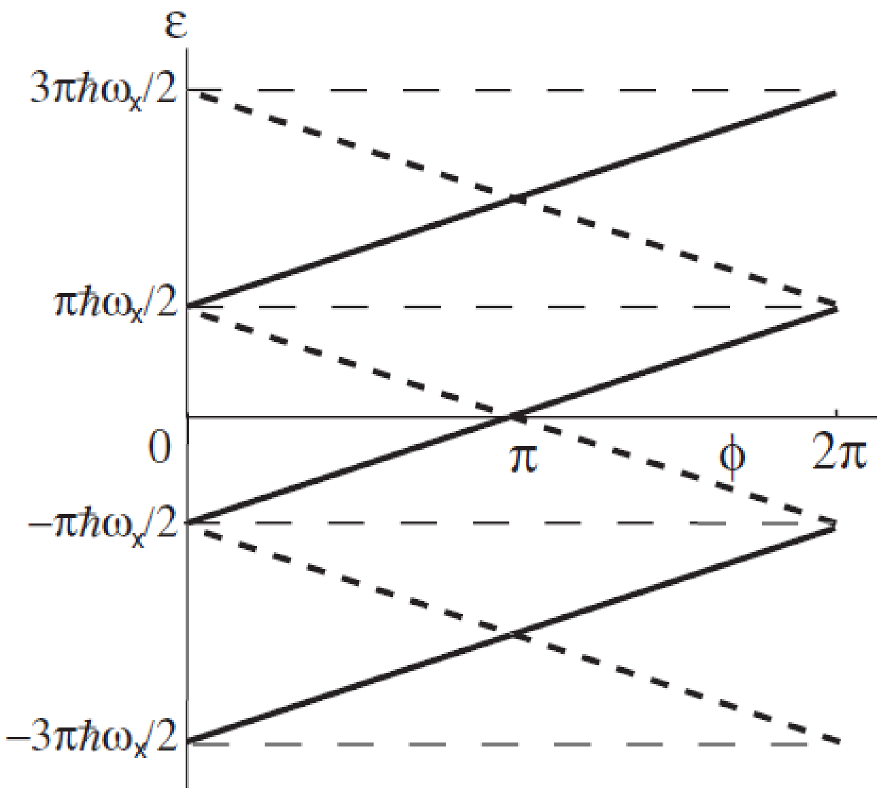
Andreev states: SNS structures

if a state with $\epsilon > 0$ belongs to $k_x > 0$ ⟹ state with $-\epsilon < 0$ belongs to $-k_x < 0$.

Short junction



Long junction



Andreev states: supercurrent through SNS

$$\mathbf{j} = -\frac{i\hbar e}{m}\sum_n [f_n (u_n^*(\mathbf{r})\nabla u_n(\mathbf{r})) + (1-f_n)(v_n(\mathbf{r})\nabla v_n^*(\mathbf{r})) - c.c]$$

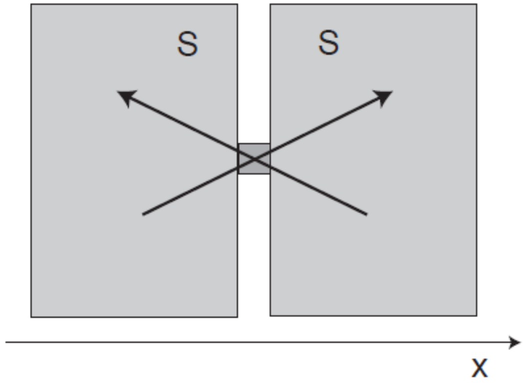
$$I_x = -\frac{e}{\hbar}\sum_n (1-2f_n)\frac{\hbar v_x\sqrt{|\Delta|^2-\epsilon_n^2}}{\hbar|v_x| + d\sqrt{|\Delta|^2-\epsilon_n^2}}$$

Current carried by the bound states

$$I_x = -\frac{e}{\hbar}\left[\sum_{n,k_x>0}(1-2f(\epsilon_>))\frac{\hbar v_x\sqrt{|\Delta|^2-\epsilon_>^2}}{\hbar v_x + d\sqrt{|\Delta|^2-\epsilon_>^2}} - \sum_{n,k_x<0}(1-2f(\epsilon_<))\frac{\hbar|v_x|\sqrt{|\Delta|^2-\epsilon_<^2}}{\hbar|v_x| + d\sqrt{|\Delta|^2-\epsilon_<^2}}\right]$$

Current carried by the continuous spectrum states $\epsilon > |\Delta|$ is zero

Case I. Short junctions.



$$\epsilon_{>,<} = \mp|\Delta|\cos\frac{\phi}{2}$$

$$I_x = \frac{eN_>|\Delta|\sin(\phi/2)}{\hbar}\tanh\frac{|\Delta|\cos(\phi/2)}{2T}$$

$N_>$ is the total number of states per unit volume with $v_x > 0$ and all possible k_y and k_z flying through the contact of an area S.

Andreev states: supercurrent through SNS. Point contact.

Point contact – is a junction where two superconductors are connected via a short and narrow constriction

$$\epsilon_{>,<} = \mp|\Delta|\cos\frac{\phi}{2} \qquad |A|^2 = \frac{\sqrt{|\Delta|^2-\epsilon^2}}{2\hbar v_F}$$

$$\begin{aligned} I_x &= -\frac{e}{\hbar}\left[\sum_{k_x>0}(1-2f(\epsilon_>))\sqrt{|\Delta|^2-\epsilon_>^2} - \sum_{k_x<0}(1-2f(\epsilon_<))\sqrt{|\Delta|^2-\epsilon_<^2}\right]\frac{|v_x|}{v_F} \\ &= \frac{e|\Delta|\sin(\phi/2)}{\hbar}\tanh\frac{|\Delta|\cos(\phi/2)}{2T}\sum_{v_x>0}\frac{v_x}{v_F} \end{aligned}$$

$$\sum_n \Rightarrow \int \frac{4\pi S p_F^2\,dp}{(2\pi\hbar)^3}\frac{d\Omega_{\mathbf{p}}}{4\pi}$$

$$p = p_F + \hbar\lambda_N = p_F + \epsilon/v_F \qquad \frac{dp}{2\pi\hbar} = \frac{d\epsilon}{2\pi\hbar v_F}C\left[\delta(\epsilon-\epsilon_>)+\delta(\epsilon-\epsilon_<)\right]$$

How to find *C*?

To find C we note that while p can be both $p < p_F$ and $p > p_F$, the energy only assumes positive values. In our case the integral $\int_{-\infty}^{\infty} dp/2\pi\hbar = 2$ since there is exactly one state per unit volume for a given phase difference ϕ for $p < p_F$ and one for $p > p_F$. Therefore,

$$\int_{\epsilon>0}\frac{d\epsilon}{2\pi\hbar v_F}C\left[\delta(\epsilon-\epsilon_>)+\delta(\epsilon-\epsilon_<)\right] = \frac{C}{\pi\hbar v_F} = 2$$

$$C = 2\pi\hbar v_F$$

Andreev states: supercurrent through SNS. Point contact.

Thus

$$\sum_n \Rightarrow 2\pi\hbar N(0) v_F S \int_{\epsilon>0} d\epsilon \frac{d\Omega_{\mathbf{p}}}{4\pi} [\delta(\epsilon - \epsilon_>) + \delta(\epsilon - \epsilon_<)]$$

Therefore

$$\sum_{v_x>0} \frac{v_x}{v_F} = 2\pi\hbar N(0) v_F S \int_{\epsilon>0} d\epsilon \int_{v_x>0} \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{v_x}{v_F} [\delta(\epsilon - \epsilon_>) + \delta(\epsilon - \epsilon_<)] = 2\pi\hbar N(0) v_F S \int_{v_x>0} \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{v_x}{v_F} = \frac{\pi\hbar N(0) v_F S}{2}$$

$$I_x = \frac{N(0) v_F S \pi e |\Delta| \sin(\phi/2)}{2} \tanh \frac{|\Delta| \cos(\phi/2)}{2T} = \frac{\pi |\Delta| \sin(\phi/2)}{e R_{\mathrm{Sh}}} \tanh \frac{|\Delta| \cos(\phi/2)}{2T}$$

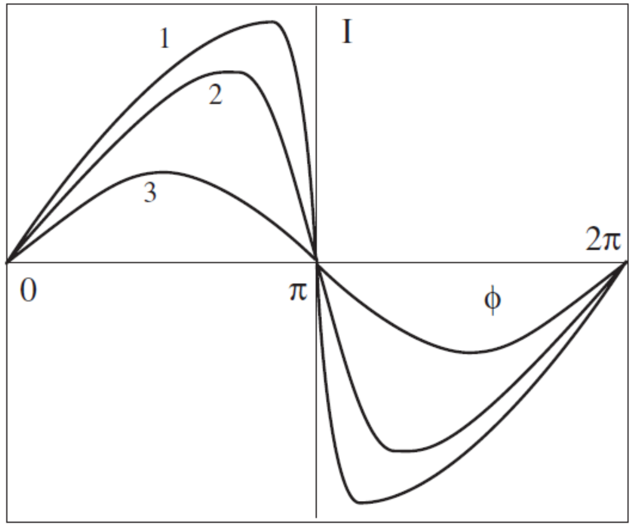
$$\frac{1}{R_{\mathrm{Sh}}} = \frac{e^2 N(0) v_F S}{2} = \frac{e^2}{\pi\hbar} \frac{\pi k_F^2 S}{(2\pi)^2}$$ - **Sharvin conductance**

$$\frac{1}{R_{\mathrm{Sh}}} = \frac{N_>}{R_0} \qquad N_> = \frac{\pi k_F^2 S}{(2\pi)^2}$$

$$R_0 = \frac{\pi\hbar}{e^2} \approx 12.9\ k\Omega$$ - quantum of resistance

Andreev states: supercurrent through SNS. Point contact.

$$I_x = \frac{\pi|\Delta|\sin(\phi/2)}{eR_{\rm Sh}}\tanh\frac{|\Delta|\cos(\phi/2)}{2T}$$



The supercurrent through the point contact. Curve (1) corresponds to a low temperature $T \ll T_c$, curve (3) is for a temperature close to T_c.

Low temperatures:

$$I_c = \frac{\pi|\Delta|}{eR_{\rm Sh}}$$

is reached at $\phi = \pi$

Temperatures close to Tc:

$$I_c = \frac{\pi|\Delta|^2}{4TeR_{\rm Sh}}$$

is reached at $\phi = \pi/2$

Andreev states: supercurrent through SNS. Long junction.

$$I_x = -\frac{e}{d}\left[\sum_{n,k_x>0} |v_x| (1-2f(\epsilon_>)) - \sum_{n,k_x<0} |v_x| (1-2f(\epsilon_<))\right]$$

$$= -\frac{e}{d}\sum_{k_x>0} |v_x| \left(\sum_{l=0}^{l_0} \tanh\left[\frac{\hbar|\omega_x|(\phi-\pi)/2+\hbar|\omega_x|\pi l}{2T}\right] - \sum_{l=1}^{l_0} \tanh\left[\frac{-\hbar|\omega_x|(\phi-\pi)/2+\hbar|\omega_x|\pi l}{2T}\right]\right)$$

$$= -\frac{e}{d}\sum_{k_x>0} |v_x| \sum_{l=-l_0}^{l_0} \tanh\left[\frac{\hbar|\omega_x|(\phi-\pi)/2+\hbar|\omega_x|\pi l}{2T}\right]$$

Here l_0 corresponds to $\epsilon = |\Delta|$, i.e., $l_0 = |\Delta|/\pi\hbar|\omega_x| \gg 1$

Consider the limit of low temperatures and very long junction $|\omega_x| \ll T \ll \Delta$ that is $d \gg \hbar v_F/T$

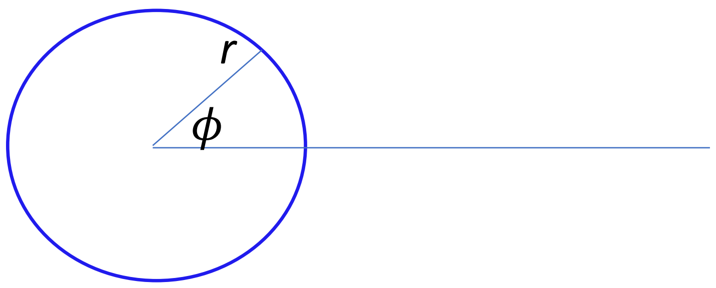
$$I_x = \frac{4\hbar N(0) v_F^2 Se}{d} e^{-d/\xi_N} \sin\phi = \frac{1}{2eR_{\rm Sh}}\frac{8\hbar v_F}{d} e^{-d/\xi_N} \sin\phi \qquad \xi_N = \frac{\hbar v_F}{2\pi T}$$

It can be written as

$I = I_c \sin\phi$ with the critical current $I_c = \frac{1}{2eR_{\rm Sh}}\frac{8\hbar v_F}{d} e^{-d/\xi_N}$

where $2R_{\rm Sh}$ is the resistance of two SN contacts in the normal state

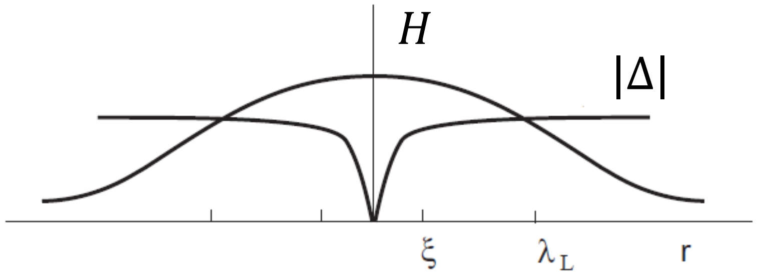
Andreev states: vortex core states.



In the cilindrical coordinate frame (r, ϕ, z) $\quad \Delta = |\Delta(r)|e^{i\phi}$

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{ik_z z} e^{i\mu\phi} \begin{pmatrix} f_+(r) e^{i\phi/2} \\ f_-(r) e^{-i\phi/2} \end{pmatrix}$$

where μ is the azimuthal quantum number. It should be the half-integer $\mu = n + 1/2$ since the wave function has to be single valued.



The BdG equations take the form:

$$-\frac{\hbar^2}{2m}\left[\frac{d^2 f_+}{dr^2} + \frac{1}{r}\frac{df_+}{dr} - \left(\frac{\mu + 1/2}{r} - \frac{eA_\phi}{\hbar c}\right)^2 f_+ + k_\perp^2 f_+\right] + |\Delta| f_- = \epsilon f_+$$

$$\frac{\hbar^2}{2m}\left[\frac{d^2 f_-}{dr^2} + \frac{1}{r}\frac{df_-}{dr} - \left(\frac{\mu - 1/2}{r} + \frac{eA_\phi}{\hbar c}\right)^2 f_- + k_\perp^2 f_-\right] + |\Delta| f_+ = \epsilon f_-$$

$$k_\perp^2 = k_F^2 - k_z^2.$$

In the limit $\lambda_L \gg \xi$.

$$\frac{eA_\phi}{\hbar c} \sim \frac{eHr}{\hbar c} \sim \frac{1}{r}\frac{H}{H_{c2}}\frac{r^2}{\xi^2} \ll \frac{1}{\xi} \qquad \text{for } r \sim \xi$$

Andreev states: vortex core states.

$$-\frac{\hbar^2}{2m}\left[\frac{d^2 f_+}{dr^2} + \frac{1}{r}\frac{df_+}{dr} - \frac{\mu^2 + 1/4 + \mu}{r^2} f_+ + k_\perp^2 f_+\right] + |\Delta| f_- = \epsilon f_+$$

$$\frac{\hbar^2}{2m}\left[\frac{d^2 f_-}{dr^2} + \frac{1}{r}\frac{df_-}{dr} - \frac{\mu^2 + 1/4 - \mu}{r^2} f_- + k_\perp^2 f_-\right] + |\Delta| f_+ = \epsilon f_-$$

Consider $|\mu| \ll k_F \xi$. Introduce r_c such that $\mu k_F^{-1} \ll r_c \ll \xi$. For $r < r_c$ we neglect $|\Delta(r)| \ll \Delta_0$. The solutions of Eqs. (3.42), (3.43) are the Bessel functions

$$f_\pm = A_\pm J_{\mu \pm 1/2}[(k_\perp \pm \lambda_N) r]$$

where $v_\perp = \hbar k_\perp / m$ and $\lambda_N = \dfrac{\epsilon}{\hbar v_\perp}$

For $r > r_c$ we look for solution in the form $\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = H_m^{(1)}(k_\perp r) \begin{pmatrix} g_+ \\ g_- \end{pmatrix} + H_m^{(2)}(k_\perp r) \begin{pmatrix} g_+^* \\ g_-^* \end{pmatrix}$

where $m = \sqrt{\mu^2 + 1/4}$ and $H_m^{(1)}$ is the Hankel function of the first kind. The amplitudes $g_\pm$ are slow function: they vary at distances of the order of ξ. For $r > r_c$ we have $dH_m^{(1)}/dr = ik_\perp H_m^{(1)}$.

Andreev states: vortex core states.

Neglecting the second derivatives of $g_{\pm}$ we obtain:

$$-\frac{i\hbar^2 k_\perp}{m}\frac{dg_+}{dr} + |\Delta| g_- = \left(\epsilon - \frac{\mu\hbar^2}{2mr^2}\right) g_+$$

$$\frac{i\hbar^2 k_\perp}{m}\frac{dg_-}{dr} + |\Delta| g_+ = \left(\epsilon - \frac{\mu\hbar^2}{2mr^2}\right) g_-$$

Look for the solution in the form:

$$\begin{pmatrix} g_+ \\ g_- \end{pmatrix} = C \begin{pmatrix} e^{i\psi(r)/2 - i\pi/4} \\ -ie^{-i\psi(r)/2 + i\pi/4} \end{pmatrix} e^{-K(r)} \quad \Longrightarrow \quad \begin{aligned} \hbar v_\perp \frac{d\psi}{dr} &= 2|\Delta|\sin\psi + 2\left(\epsilon - \frac{\mu\hbar^2}{2mr^2}\right) \\ \hbar v_\perp \frac{dK}{dr} &= |\Delta|\cos\psi \end{aligned}$$

We shall see that for $\mu k_F^{-1} \ll \xi$, the function ψ is small. Therefore,

$$K(r) = (\hbar v_\perp)^{-1}\int_0^r |\Delta(r')|\, dr'$$

$$\psi(r) = -e^{2K(r)}\int_r^\infty \left(2\lambda_N - \frac{\mu}{k_\perp r'^2}\right) e^{-2K(r')}\, dr'$$

The constant of integration here is taken to make ψ a bounded function for $r \to \infty$. The second term under the integral diverges for $r \to 0$. Integrating by parts, we obtain

$$\psi(r) = \frac{\mu}{k_\perp r} + 2\lambda_N r - 2e^{2K(r)}\int_0^\infty \left(\lambda_N + \frac{\mu|\Delta(r)|}{\hbar k_\perp v_\perp r}\right) e^{-2K(r)}\, dr$$

The term under the integral has now no singularities.

Andreev states: vortex core states.

Match the solutions at $r = r_c$

$$x \gg |\alpha^2 - 1/4| \qquad J_\alpha(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)$$

$$J_{\mu\pm 1/2}[(k_\perp \pm \lambda_N) r_c] = [2/\pi k_\perp r_c]^{1/2} \cos\left[(k_\perp \pm \lambda_N) r_c + \frac{(\mu \pm 1/2)^2}{2k_\perp r_c} - \frac{\pi}{2}\left(\mu \pm \frac{1}{2}\right) - \frac{\pi}{4}\right]$$

$$H_m^{(1)}(k_\perp r_c) = [2/\pi k_\perp r_c]^{1/2} \exp\left[i\left(k_\perp r_c + \frac{m^2}{2k_\perp r_c} - \frac{\pi m}{2} - \frac{\pi}{4}\right)\right]$$

Since $2J(x) = H^{(1)}(x) + H^{(2)}(x)$ the matching requires

$$k_\perp r_c + \frac{m^2}{2k_\perp r_c} - \frac{\pi m}{2} - \frac{\pi}{4} + \frac{\psi(r_c)}{2} - \frac{\pi}{4} = (k_\perp + \lambda_N) r_c + \frac{(\mu + 1/2)^2}{2k_\perp r_c} - \frac{\pi}{2}\left(\mu + \frac{1}{2}\right) - \frac{\pi}{4}$$

For $\mu \gg 1$ when $m = \mu$ this gives $\displaystyle\int_0^\infty \left(\lambda_N + \frac{\mu|\Delta(r)|}{\hbar k_\perp v_\perp r}\right) e^{-2K(r)}\, dr = 0$

$$\epsilon_\mu(k_z) = -\mu k_\perp^{-1} \frac{\int_0^\infty (|\Delta(r)|/r)\, e^{-2K(r)}\, dr}{\int_0^\infty e^{-2K(r)}\, dr}$$

Andreev states: vortex core states.

Localized states $\epsilon_\mu(k_z) = -\mu k_\perp^{-1} \frac{\int_0^\infty (|\Delta(r)|/r)\, e^{-2K(r)}\, dr}{\int_0^\infty e^{-2K(r)}\, dr}$ form an equidistant spectrum:

$$\epsilon_\mu = -\mu\omega_0(k_z)$$

where the interlevel spacing

$$\omega_0 \sim \frac{\Delta_0}{p_F \xi} \sim \frac{\Delta_0^2}{E_F} \ll \Delta_0$$

Caroli, de Gennes, Matricon, 1964

The energy spectrum holds for $\mu \ll k_F \xi$.

$k_F \xi \sim E_F/\Delta_0 \gg 1$ ➡ $\epsilon \ll \Delta_0$

For larger μ energy approaches $\mp\Delta_0$

Since $\mu = n + 1/2$ the lowest energy is nonzero: there exists a minigap $\omega_0/2$

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Reconstruction of the Density of States at the End of an S/F Bilayer

I. V. Bobkova & A. M. Bobkov

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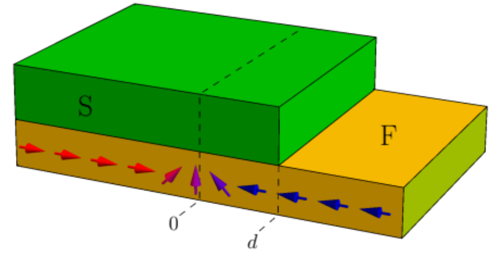
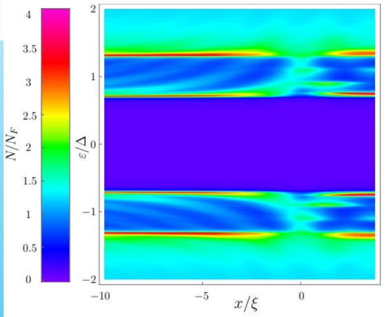
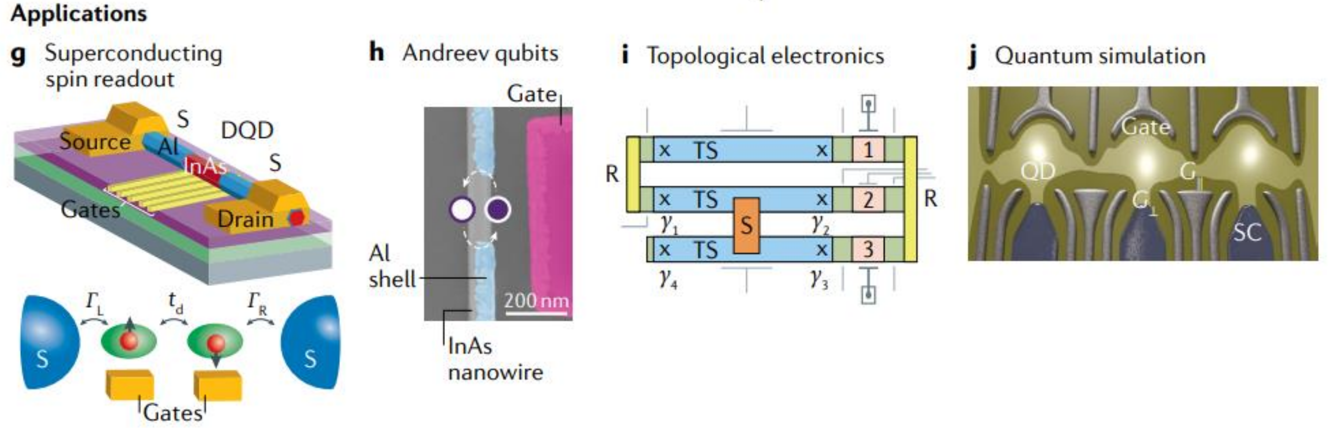




From Andreev to Majorana bound states in hybrid superconductor–semiconductor nanowires

Elsa Prada[1], Pablo San-Jose[2], Michiel W. A. de Moor[3], Attila Geresdi[3], Eduardo J. H. Lee[1], Jelena Klinovaja[4], Daniel Loss[4], Jesper Nygård[5], Ramón Aguado[2] and Leo P. Kouwenhoven[3,6]



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Observation of ordered vortices with Andreev bound states in $Ba_{0.6}K_{0.4}Fe_2As_2$

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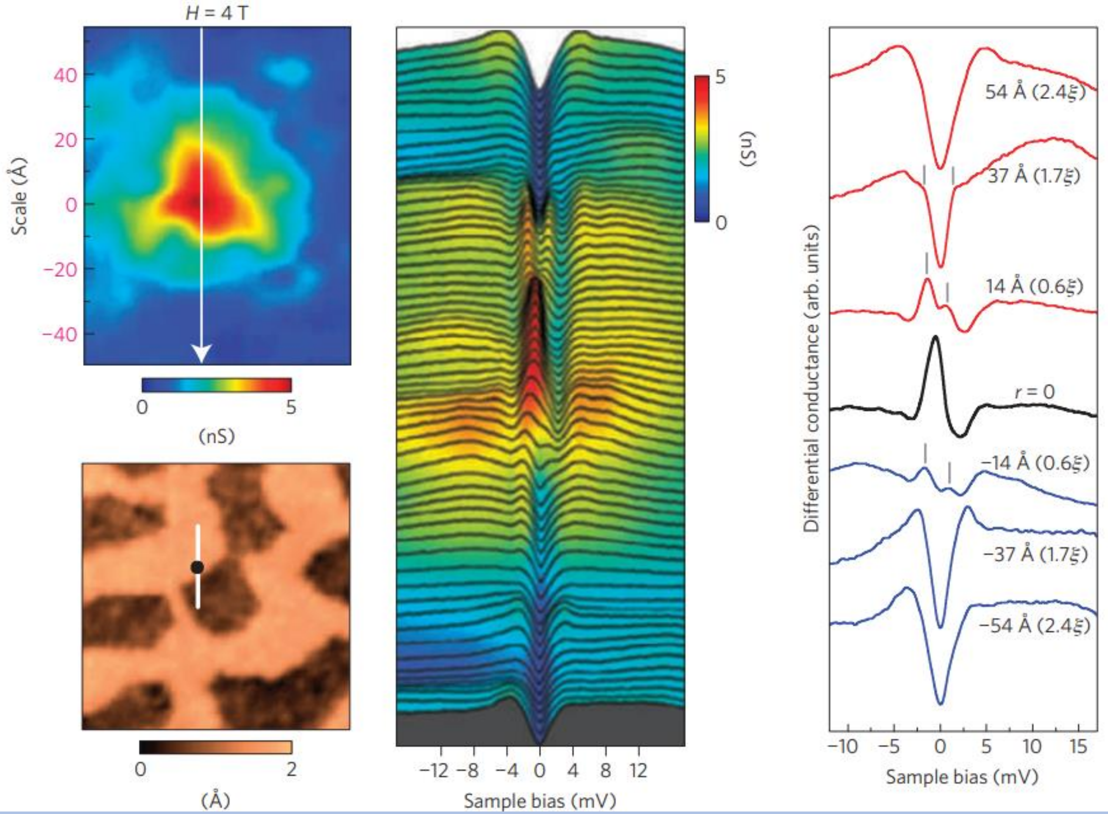

Problems to Section 2

2.1 Find the deflection angle of the trajectory during Andreev reflection process

2.2. Derive the expression for Andreev bound state energy spectrum in SNS structure at $k_x < 0$

2.3. Find the energy spectrum and the wave functions of the short SNS junction at $\epsilon < |\Delta|$

2.4 Find the wave functions for an SNS structure at $\epsilon > |\Delta|$
Prove that these energies do not contribute to the Josephson current through the SNS structure